Timoshenko beam and plate non-stationary vibrations

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Abstract: *The problems of Timoshenko beams and plates lateral vibrations under the influence of unsteady loads are considered. Both beam and plate is supposed to be unlimited. In case of the plate the problem has been simply studied. The approach to the solution was based on dominant function method and principle of superposition. Integral models of solutions with cores as dominant functions were built which could be analytically found with the help of the Fourier and Laplace integral transforms. Two original analytical methods for Fourier and Laplace transforms were offered and realized. The examples of calculations were given.*

Key Words*: Timoshenko plate, Timoshenko beam, non-stationary vibrations, superposition method, influence functions, Fourier series, integral transforms*

1. INTRODUCTION

Now, non-stationary mechanical problems of the deformable solid become more and more actual in theoretical and extra spheres. It relates to the increasing demands to the mathematical accuracy of stress and bearing capacity of the main engineering elements, working under the influence of unsteady loads. The study of dynamical processes in beams and plates is actual for many industrial spheres. It's due to the wide application of thin-walled elements of beams and plates type in models of aerospace craft, cars, ships, different engineering and building structures. Recently, due to the rapid development of modern technologies and growing demands for the exactness of non-stationary stress-strain state and safety factor, the increasing attention is paid to the non-stationery mechanical tasks of thin walled structures. It's worth mentioning that nevertheless now, there is a very few works devoted to these problems. We may note the following works related to the theme of this article. In this article a numerical-analytical technique is used to solve the problem of non-stationary cross vibrations for

Timoshenko beam as well as flat and axially-symmetrical non-stationary problem on the movement of unlimited Timoshenko-type plate.

It's based on the application of method of influence functions, where first of all fundamental solutions (influence functions) are built by analytical methods with for beam and plate. Then, based on the superposition principle this problem relates to the permissive integral ratio. It, in many cases, allows getting an approximate solution with analytical methods or build highly-efficient numerical-analytical calculation algorithms. The formed technique can serve as the base for the development of mathematical statements and methods for solution of new non-stationary contact problems as well as non-stationary inverse problems of beams and plates [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. The formed technique can serve as the base for the development of mathematical statements and methods for solution of new non-stationary contact problems as well as non-stationary inverse problems of beams and plates.

2. MATERIALS AND METHODS

Within the work [13] they attempted to describe a mathematical theory of vibrations for elastic or viscoelastic plate for studying of its dynamical behavior under non-stationary external influences. Based on such approach exact equations of longitudinal and lateral vibrations of viscoelastic plates were derived considering and without taking into account its initial displacements and stresses, approximate equations of physical nonlinearity.

The work [14] investigated the postbuckling for free-standing thin-walled beam lying on Winkler-Pasternak elastic base and being under axial compression load. It's supposed that the beam material is elastic, beam deflections are not significant. Beam transverse shear deformation and cross section deformations can be neglected. The postbuckling for free-standing thin-walled beam under different Winkler and Pasternak foundation values was calculated. Bifurcation points were found.

The article [15] includes the statement of boundary value problem on the axially symmetric elastic thr ee-layer circular plate on Pasternak foundation that allows considering the influence of base shear properties on stress-strain state of calculating structure.

The Broken Line hypothesis was adopted to describe the kinematics of asymmetric plate. Kirchhoff's hypothesis on standard incompressibility, rectilinearity and perpendicularity to deformed middle surface is suitable for thin carriers. In relatively thick incompressible filler Timoshenko hypothesis is fulfilled with linear approximation of displacements over the layer width.

The article [16] describes the problem on the growth of non-stationary boundary bending wave over the thin plate. The elastic boundary wave in free thin half-infinite isotropic plate was originally investigated by Y.K. Konenkov in his work. One of the most important characteristics of Konenkov wave is its dispersion, i.e. dependence of wave speed from its own plate frequencies.

The article [17] describes the method for solving of some specific problems concerning the elastic beam vibrations in incompressible non-stationary flow using MSC Adams and external module simulating non-stationary hydrodynamic flow due to incompressible fluid of deforming moving body.

The article [18] describes the method of calculation of non-stationary vibrations for multilayer plates under pulse loading. The influence of aggregate elastic module was investigated and loading localization on stresses in sandwich plates. A comparative analysis concerning calculation data based on different theories of sandwich structures was carried out.

The forced vibrations for circular plate growing per width in case of small deformations were investigated in the article [19]. It’s recognized that the plate material is elastic and isotropic and its width is constantly growing as a result of external material inflow. It’s supposed that the width of the growing plate varies with time but doesn’t depend from space coordinates. More over in the process of growth the middle surface position doesn’t grow i.e. plate increasing occurs symmetrically on both face surfaces.

The article [20] is devoted to the investigation of non-stationary vibrations for thin rectangular plate “metal-piezoceramics” type under mechanical loading. Using superposition method and Laplace integral transformation with time the initial boundary value is related to Volterra integral equation. A technique for solving the problem of mechanical loading identification as time-varying function per potential values between continuous electrodes of piezoelectric element is also represented. The article [21] describes non-stationary deformation of medium rectangular plate with fixed vibration damper. It’s supposed that the force acts transversely and the damper influence is registered as unknown point load. The problem is studied within the Timoshenko-type theory and relates to the solution of Volterra integral equation of the I kind.

Methods for solving non-stationary calculating the vibrations of membrane under lateral load are studied in article [22]. Such methodic includes the use of Laplace and Fourier integral transformations. The data found reliability is discussed.

In work [23] the solution of non-standard dynamical problem is given for the plate deforming in accordance with Timoshenko hypothesis in arbitrary curvilinear orthogonal coordinates. The solution is formed in series on its own forms based on found conditions of orthogonality of its own forms and is valid for any combination of the following conditions on plate contours: free edge, hinged edge and rigid termination. Some computational results are given in pulse loading of free ring plate.

The article [24] is devoted to the problem solution of vibrations for the infinite elastic plate lying on elastic isotropic half-space. Non-stationary vibrations are caused by the influence of moment loads on the plate leading to the appearance on the elastic half-space of two types of flat broken surfaces, beyond its fronts up to the contact boundary the solution is formed with the help of ray rows. The unknown functions included in ray rows and plate vibration equation are indicated based on boundary conditions of contact plate interaction with half-space.

The article [25] describes the problem of vibrations for standard infinite elastic plates, covering the boundary of the anisotropic elastic half-space. It’s supposed that there is no friction between the plate and half-space boundary, and continuous normal and tangential forces are applied. Non-stationary vibrations are caused by the influence of dynamic loads on the plate leading to the appearance of three types of flat dynamic waves in the elastic anisotropic half-space, beyond its borders the solution is formed with the help of ray rows.

The articles [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40] and [20] describe the mathematical statements, methods and approaches to the solution of different non-stationary contact problems concerning the mechanics of deformable solids, including non-stationary contact problems for thin-walled structural elements. These works successfully use the influence function method (Green’s functions) to solve different non-stationary problems.

3. RESULTS AND DISCUSSIONS

Let’s study the statement of the problem on non-stationary vibrations of Timoshenko beam. Suppose that at any initial time the beam, unlimited in length, rests that corresponds to zero-initial conditions.To describe the beam movement a set of motion equations is used for Timoshenko beam (the beam axis coincides with the axis Ox) [41]:

$$\frac{\partial^2 w}{\partial t^2} = k^2 c_2^2 \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} - \chi\right) + \frac{p}{\rho F}, \frac{\partial^2 \chi}{\partial t^2} = c_p^2 \frac{\partial^2 \chi}{\partial x^2} - \frac{c_2^2 F k^2}{I}\left(\chi - \frac{\partial w}{\partial x}\right). \quad (1)$$

where w – beam deflection, F – sectional area, ρ – density of the material, $c_2 = \sqrt{\frac{G}{\rho}}$ – shear wave velocity, $c_p = \sqrt{\frac{E}{\rho}}$ – speed of bending waves, E and G – Young's module and shear module, $k^2 = \frac{5}{6}$ – correction factor of shear, χ – cross section angle due to shear deformations, I – moment of inertia of cross section towards the main central axis, t – time, $p(x,t)$ – lateral load.

Let's eliminate the system of dimensionless quantities (dimensional parameters are marked with dashed line):

$$\tau = \frac{c_2 t}{L},\ w = \frac{w'}{L}, x = \frac{x'}{L}, \eta = \frac{c_p}{c_2}, L = \sqrt{\frac{I}{F}}, p = \frac{p' L}{\rho F c_2^2} \quad (2)$$

Then, in dimensionless form the equations (1) would be as follows:

$$\frac{\partial^2 w}{\partial \tau^2} = k^2\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \chi}{\partial x}\right) + p, \frac{\partial^2 \chi}{\partial \tau^2} = \eta^2 \frac{\partial^2 \chi}{\partial x^2} - k^2\left(\chi - \frac{\partial w}{\partial x}\right) \quad (3)$$

Zero-initial conditions add the set (3):

$$w|_{\tau=0} = \dot{w}|_{\tau=0} = \chi|_{\tau=0} = \ddot{\chi}|_{\tau=0} = 0 \quad (4)$$

Note that described above statement of problem on non-stationary vibrations of Timoshenko beam is close to the problem on Timoshenko plate vibrations in case of flat statement. Actually, if we study the equation of unlimited Timoshenko plate in Cartesian rectangular coordinate system Oxy, which Cartesian plane coincides with the middle plane of the plate and suppose that the spreading of surface pressure does not depend from the coordinate y: $p(x,y,\tau) = p(x,\tau)$, then in dimensionless values:

$$\tau = \frac{c_2 t}{L},\ w = \frac{w'}{L}, x = \frac{x'}{L}, \eta = \frac{c_1}{c_2}, L = \sqrt{\frac{I}{h}}, p = \frac{p' L}{\rho h c_2^2}, \quad (5)$$

where $c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ – speed of tensile-compression waves, λ, $\mu = G$ – Lame parameters, $I = \frac{h^3}{12}$, h – plate thickness, come to the following equations [41; 42]:

$$\frac{\partial^2 w}{\partial \tau^2} = k^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \chi}{\partial x}\right) + p, \frac{\partial^2 \chi}{\partial \tau^2} = \eta^2 \frac{\partial^2 \chi}{\partial x^2} - k^2\left(\chi + \frac{\partial w}{\partial x}\right) \quad (6)$$

which in the sense of correspondence of differential operators are analogous to the equations (3). Up to the closed statement of the problem they can be also added by initial equations (4). It's required to calculate the beam or plate deflection $w(x,\tau)$ under the influence of distributed transient load $p(x,\tau)$.

The method of solution is based on the principle of superposition [26], [27], [28], [29], [30], and [20]. Besides, the desired solutions represent a set of given load with the influence function for beam or plate on space coordinates and time. For influence functions let's input the following designations: influence function for Timoshenko beam – $G_b(x,\tau)$, for Timoshenko plate in case of flat statement of the problem – $G_p(x,\tau)$, for Timoshenko plate in case of axially symmetrical statement of the problem – $G_o(r,\tau)$. The influence functions represent standard displacements of the beam or plate, as problem solutions (3), (4) or (6), (4) or Error, Link Source not found (4) when setting the following functions as loads: $p=\delta(x)\delta(\tau)$ – in case of beam or plate in flat statement and $p=\delta(x,y)\delta(\tau)$ – in case of plate in axially symmetrical statement.

If influence functions are known then the required solutions are immediately recorded in quadrature [32], [33], [34], [35], [36], [37], [38], [39], [40] and [20]:

- for Timoshenko beam:

$$w(x,\tau)=\int_0^\tau\int_{-\infty}^{\infty}G_b(x-\xi,\tau-t)p(\xi,t)\,d\xi dt; \tag{7}$$

- for Timoshenko plate in case of flat statement of the problem we have the analogous integral approximation:

$$w(x,\tau)=\int_0^\tau\int_{-\infty}^{\infty}G_p(x-\xi,\tau-t)p(\xi,t)\,d\xi dt; \tag{8}$$

We can set the problem on the influence function for Timoshenko beam:

$$\begin{aligned}&\frac{\partial^2 G_b}{\partial\tau^2}=k^2\left(\frac{\partial^2 G_b}{\partial x^2}-\frac{\partial\chi}{\partial x}\right)+\delta(x)\delta(\tau),\frac{\partial^2\chi}{\partial\tau^2}=\eta^2\frac{\partial^2\chi}{\partial x^2}-k^2\left(\chi-\frac{\partial G_b}{\partial x}\right),\\&G_b|_{\tau=0}=\dot G_b\big|_{\tau=0}=\chi|_{\tau=0}=\ddot\chi|_{\tau=0}=0.\end{aligned} \tag{9}$$

To solve such problem we may use Laplace integral transformation per time and Fourier per coordinate x (s – Laplace transformation parameter, q – Fourier transformation parameter):

$$f^{FL}(q,s)=\int_0^\infty\int_{-\infty}^{\infty}f(x,\tau)e^{iqx+s\tau}dxd\tau \tag{10}$$

Hereinafter the upper sign F means its Fourier transformation and L – Laplace transformation.

Applying (10) to (9) and considering that $[\delta(x)\delta(\tau)]^{FL}=1$, we may get:

$$(s^2+k^2q^2)G_b^{FL}-iqk^2\chi^{FL}=1,iqk^2G_b^{FL}+(s^2+\eta^2q^2+k^2)\chi^{FL}=0 \tag{11}$$

Here we may find the description of influence function for Timoshenko beam:

$$G_b^{FL}=\frac{s^2+\eta^2q^2+k^2}{(s^2+k^2q^2)(s^2+\eta^2q^2)+k^2s^2} \tag{12}$$

The creation of function original (12) represents any specific difficulties, because it's not impossible to fulfill any consistent handling. There is also no possibility to use the method of joint inversion for integral transformations [41], [42] since both the numerator and denominator of equation (12) are not homogenous functions [43], [44], [45], [46], [47].

To build the original (12) we may suppose to use two approaches. The first one includes the expansion (12) on degrees of expression $\frac{k^2s^2}{(s^2+k^2q^2)(s^2+\eta^2q^2)} < 1$:

$$G_b^{FL} = \frac{s^2+\eta^2q^2+k^2}{(s^2+k^2q^2)(s^2+\eta^2q^2)}\sum_{n=0}^{\infty}(-1)^n\frac{k^{2n}s^{2n}}{(s^2+k^2q^2)^n(s^2+\eta^2q^2)^n} = \\ = \sum_{n=0}^{\infty}(-1)^n\frac{(s^2+\eta^2q^2+k^2)k^{2n}s^{2n}}{(s^2+k^2q^2)^{n+1}(s^2+\eta^2q^2)^{n+1}}. \tag{13}$$

The original of each term of the series (13) can be analytically found with the help of residue theorem.

$$\left[(-1)^n\frac{(s^2+\eta^2q^2+k^2)s^{2n}k^{2n}}{(s^2+k^2q^2)^{n+1}(s^2+\eta^2q^2)^{n+1}}\right]^{L^{-1}} = \\ = \frac{(-1)^n}{n!}\sum_{m=1}^{4}\boldsymbol{\lim_{s\to s_m}}\frac{d^n}{ds^n}\left[\frac{(s^2+\eta^2q^2+k^2)s^{2n}k^{2n}(s-s_m)^{n+1}}{(s^2+k^2q^2)^{n+1}(s^2+\eta^2q^2)^{n+1}}e^{s\tau}\right], s_{1,2} \\ = \pm ik|q|, s_{3,4} = \pm i\eta|q| \tag{14}$$

Here L^{-1} means the inverse Laplace transformation.

The second method is based on the Fourier integral relationship with Fourier series on variable interval. We may suppose that the distribution process for non-stationary vibrations in the sphere can be described by a specific hyperbolic equation or set of equations towards the function $f(x,\tau)$ (or one of the searching functions in case of set of equations). Since the initial equations have a hyperbolic type, then the speed of distribution for vibrations $f(x,\tau)$ will be definite and, as general, its significance may be indicated directly knowing the coefficients of the initial equations (or equations characteristics). We assume that the speed of vibration distribution $f(x,\tau)$ in the sphere is equal to c.

Then, if the carrier of the external load is concentrated in point $x = 0$, that corresponds to the statement of problem on the influence function then we know that the carrier of perturbations $f(x,\tau)$ is the following set $\{x: |x| \le c\tau = l(\tau)\}$, i.e. $f(x,\tau) \equiv 0$ when $|x| > c\tau$. Study the expansion of a function in Fourier series $f(x,\tau)$ on interval $|x| < L$ believing that $L > l(\tau)$:

$$f(x,\tau) = \frac{1}{2L}\sum_{m=-\infty}^{\infty}\left[e^{-i\frac{\pi m x}{L}}\int_{-L}^{L} f(x,\tau)e^{i\frac{\pi m x}{L}}dx\right] \tag{15}$$

Remembering that $f(x,\tau)$ has the limited carrier $|x| < l(\tau) \le L$, integrals in (15) may be recorded as follows:

$$\int_{-L}^{L} f(x,\tau)e^{i\frac{\pi m x}{L}}dx = \int_{-\infty}^{\infty} f(x,\tau)e^{i\frac{\pi m x}{L}}dx = f^F\left(\frac{\pi m}{L},\tau\right) \tag{16}$$

So, the coefficients of series (15) coincide with the values of Fourier transformation in points $\frac{\pi m}{L}$:

$$f(x,\tau)=\frac{1}{2L}\sum_{m=-\infty}^{\infty} f^F\left(\frac{\pi m}{L},\tau\right)e^{-i\frac{\pi m x}{L}} \tag{17}$$

The value of decomposition interval $2L$ may be randomly selected but considering any limits $l(\tau)\leq L$. To increase the coincidence we may input $L=l(\tau)$, that will lead to the series with decomposition interval:

$$f(x,\tau)=\frac{H[l(\tau)-|x|]}{2l(\tau)}\sum_{m=-\infty}^{\infty} f^F\left[\frac{\pi m}{l(\tau)},\tau\right]e^{-i\frac{\pi m x}{l(\tau)}} \tag{18}$$

where H– Heaviside function.

So, the equation (18) may serve for the calculation of Fourier inverse transformation. We may note that the series sum (18) does not depend from change $L=l(\tau)$, consequently, in the process of series (18) term-by-term differentiation or integration we may assume $l(\tau)=\text{const}$.

The ratio (18) may be used also for the inverse of integral Fourier transformation together with the inverse of Laplace transformation. Executing direct and inverse Laplace transformation over the equation (18) and assuming that $l(\tau)=\text{const}$, we may get:

$$f(x,\tau)=\frac{H[l(\tau)-|x|]}{2l(\tau)}\sum_{m=-\infty}^{\infty} f^{FL^{-1}}\left[\frac{\pi m}{l(\tau)},s\right]e^{-i\frac{\pi m x}{l(\tau)}} \tag{19}$$

Note that if $f(x,\tau)$ – even function on variable x: $f(-x,\tau)=f(x,\tau)$, then its Fourier exponential transform on x coincides with Fourier cosine transformation and instead of (19) we may use the following equation

$$\begin{gathered}f(x,\tau)=\frac{1}{2l(\tau)}f^{FL^{-1}}(0,s)H[l(\tau)-|x|]\\+\frac{1}{l(\tau)}\sum_{m=1}^{\infty}\left\{f^{FL^{-1}}\left[\frac{\pi m}{l(\tau)},s\right]\boldsymbol{cos}\left[\frac{\pi m x}{l(\tau)}\right]\right\}H[l(\tau)-|x|].\end{gathered} \tag{20}$$

The set of equations (3) (or (6)) has a hyperbolic type [41]. Besides, maximum dimensionless wave propagation velocity is equal to $c\,max(_{max}$. On this assumption, when constructing the inverse transformations for (12) with the help of equation (19) we may get $l(\tau)=c_{max}$.

Study the equation $G_b^{FL}(q,s)$. Find the roots of the denominator s_k, $k=\overline{1,4}$ denominator $(s^2+k^2q^2)(s^2+\eta^2q^2)+k^2s^2$:

$$\begin{gathered}s_{1,2}=\pm i\alpha, s_{3,4}=\pm i\beta, \alpha=\alpha(q)=\frac{1}{\sqrt{2}}\sqrt{b-\sqrt{D}}, \beta=\beta(q)=\frac{1}{\sqrt{2}}\sqrt{b+\sqrt{D}}, b\\=b(q)=q^2(\eta^2+k^2)+k^2>0, D=D(q)\\=[q^2(\eta-k)^2+k^2][q^2(\eta+k)^2+k^2]>0\end{gathered} \tag{21}$$

All roots s_k – simple and only visible, because:

$$b^2-D=4\eta^2k^2q^4>0 \;\Rightarrow\; b>\sqrt{D} \tag{22}$$

Using the second decomposition theorem for Laplace transformation we may execute the inverse Laplace transform of function $G_b^{FL}(q,s)$:

$$G_b^F(q,\tau)=[G^{FL}(q,s)]^{L^{-1}}=\sum_{k=1}^{4}A_k(q)e^{s_k\tau},A_k(q)=(s-s_k)G^{FL}(q,s_k) \tag{23}$$

Noting that the denominator in equation for the function $G_b^{FL}(q,s)$ may be represented as:

$$(s^2+k^2q^2)(s^2+\eta^2q^2)+k^2s^2=\prod_{k=1}^{4}s-s_k=(s-i\alpha)(s+i\alpha)(s-i\beta)(s+i\beta) \tag{24}$$

Write out in details the expressions for $A_k(q)$:

$$A_1(q)=\frac{-\alpha^2+\eta^2q^2+k^2}{2i\alpha(i\alpha-i\beta)(i\alpha+i\beta)}=-\frac{-\alpha^2+\eta^2q^2+k^2}{2i\alpha(\alpha-\beta)(\alpha+\beta)}=\frac{1}{2i}f_1(q) \tag{25}$$

$$f_1(q)=\frac{\alpha^2-\eta^2q^2-k^2}{\alpha(\alpha^2-\beta^2)} \tag{26}$$

$$A_2(q)=\frac{-\alpha^2+\eta^2q^2+k^2}{-2i\alpha(-i\alpha-i\beta)(-i\alpha+i\beta)}=\frac{-\alpha^2+\eta^2q^2+k^2}{2i\alpha(\alpha^2-\beta^2)}=-\frac{1}{2i}f_1(q) \tag{27}$$

$$A_3(q)=\frac{-\beta^2+\eta^2q^2+k^2}{2i\beta(i\beta-i\alpha)(i\beta+i\alpha)}=\frac{-\beta^2+\eta^2q^2+k^2}{2i\beta(\alpha^2-\beta^2)}=\frac{1}{2i}f_2(q) \tag{28}$$

$$f_2(q)=\frac{-\beta^2+\eta^2q^2+k^2}{\beta(\alpha^2-\beta^2)}, \tag{29}$$

$$A_4(q)=\frac{-\beta^2+\eta^2q^2+k^2}{-2i\beta(-i\beta-i\alpha)(-i\beta+i\alpha)}=-\frac{-\beta^2+\eta^2q^2+k^2}{2i\beta(\alpha^2-\beta^2)}=-\frac{1}{2i}f_2(q), \tag{30}$$

Considering the last calculations formula (23) may be rewritten as:

$$G_b^F(q,\tau)=f_1(q)\frac{e^{i\alpha\tau}-e^{-i\alpha\tau}}{2i}+f_2(q)\frac{e^{i\beta\tau}-e^{-i\beta\tau}}{2i}= \\ =f_1(q)\,\boldsymbol{sin}\,\alpha\,\tau+f_2(q)\,\boldsymbol{sin}\,\beta\,\tau. \tag{31}$$

Note that the searching original $G_b(x,\tau)$ is even function on variable x, so the original of influence function may be built on formula (20), supposing $l(\tau)=c_{max}$. Preliminary find:

$$b(0)=k^2,D(0)=k^4,\alpha(0)=0,\beta(0)=k,f_2(0)=0\ \boldsymbol{lim}_{\alpha\to0}f_1(q)=\infty,f_1(q) \\ \sim\frac{1}{\alpha}\ (\alpha\to0),\boldsymbol{lim}_{\alpha\to0}f_1(q)\,\boldsymbol{sin}\,\alpha\,\tau=\boldsymbol{lim}_{\alpha\to0}\frac{\boldsymbol{sin}\,\alpha\,\tau}{\alpha}=\tau. \tag{32}$$

Considering (31), (32) on formula (33) we may get the expression for the original of influence function $G_b(x,\tau)$:

$$G_b(x,r)=\frac{1}{2c_{\max}}H\left(C_{\max r-|x|}\right)+\left(\frac{H(c\max r-|x|)}{c\max r}\right)$$
$$\cdot\sum_{m=1}^{\infty}(f_1\sin\alpha_m r+f_2\sin\beta_m r)\cos\frac{\pi m x}{c\max r},f_1 m$$
$$=f_1\left(\frac{\pi m}{C\max\tau}\right),f_{2m}=f_2\left(\frac{\pi m}{c\max r}\right),\alpha_m=\alpha\left(\frac{\pi m}{c\max r}\right),\beta_m$$
$$=\beta\left(\frac{\pi m}{c\max r}\right) \tag{33}$$

Performing calculations for formula (33) it's necessary to be limited by the finite number of terms taking into account the analysis of practical convergence [48], [49], [50], [51]. Executing Fourier integral transformation on coordinate x and Laplace per time over equations (6) considering initial conditions (4) and $p(x,\tau)=\delta(x)\delta(\tau)$, we may find the image on Fourier and Laplace of the influence function for Timoshenko plate:

$$G_p^{FL}=\frac{s^2+\eta^2q^2+k^2}{(s^2+q^2k^2)(s^2+\eta^2q^2)+k^2s^2}=G_b^{FL} \tag{34}$$

It's clear that the image of the influence function for Timoshenko plate exactly coincides with the image of the influence function for Timoshenko beam. Such result seems to be strange is quite explainable. Sets of equations (3) and (6) differ only by the signs of the first derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial \chi}{\partial x}$ and in case of change $x=-x$ pass one into the other. And since the deflection is even function then the fact of equality for the influence functions $G_b(x,\tau)$ and $G_p(x,\tau)$ also follows from it. So, further we will not differentiate it and will designate as follows $G_b(x,\tau)=G_p(x,\tau)=G(x,\tau)$.

In view of the above said we restrict ourselves to considering only Timoshenko beam. Steel will serve as beam material with the following characteristics: $\rho=7850$ kg/m^3, $E=2\cdot 10^{11}$ N/m^2, $\nu=0.3$ (Poison's ratio). The correspondent dimensionless parameter $\eta=1.61$. In Figure 1 graphs of the original influence function are represented built with the help of the first offered method depending on x at a time $\tau=3$ considering different number of terms in series (14).

In Figure 1.a the solid curve corresponds only to the first term in the series (14), dashed line – the first three terms and dashed dotted line –the first five members. In Figure 1b the analogous curves correspond to the accounting of the first three, five and ten members in row (14) [52], [53].

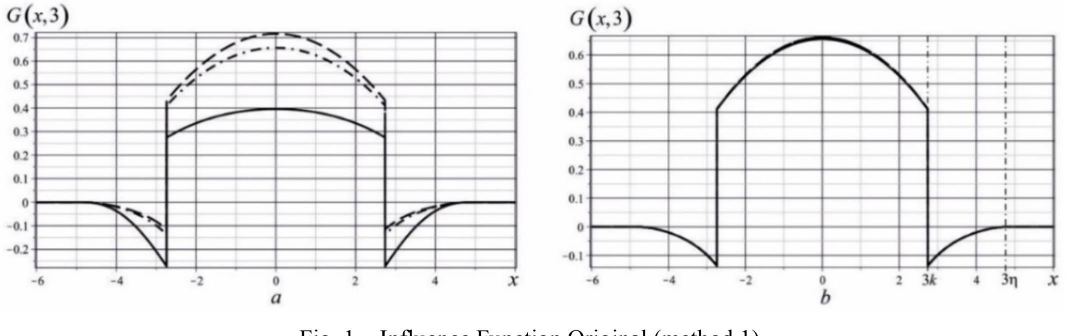

Fig. 1 – Influence Function Original (method 1)

Considering five or ten members the ratio error estimated following the Chebyshev norm $\|f(x)\| = \max_{x\in(-\infty,\infty)} |f(x)|$ was:

$$\frac{\|G_5(x,3) - G_{10}(x,3)\|}{\|G_5(x,3)\|}100\% = 0.018\%, \tag{35}$$

where $G_5(x,3)$ – influence function considering the first five members and $G_{10}(x,3)$ – influence function considering the first ten members.

The analysis of the results reveals the character of wave processes in Timoshenko beam and plate. It can be seen that the wave process can be characterized by the presence of two wave fronts.

A shear wave propagates ahead with a dimensionless speed η. Due to the connectability of sets of movement equations it immediately generates a bending wave.

Then, the bending wave follows with dimensionless speed k where $x = k\tau$ a discontinuity of the first kind is available. The position of fronts is shown in fig. 1.b by vertical dash-dotted lines.

In Figure 2 graphs of original influence function are represented. It is built with the help of the second offered method depending on the coordinate x at time $\tau = 3$ considering different number of terms in series (33).

In Figure 1 a block curve corresponds to the account of the first five memebers in series (33), dash line corresponds to ten members and dash-dotted corresponds to twenty first members. In fig. 1.b the analogous curves correspond to the account of forty, sixty and one hundred members in series (33).

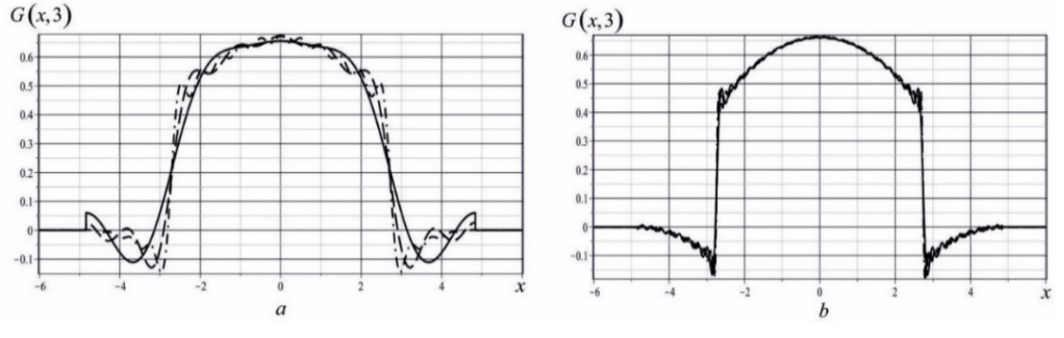

Fig. 2 – Influence function original (method 2)

The ratio error accounting sixty and one hundred members in series (33), estimated according to norm in space $L_2(-\infty,\infty)\|f(x)\| = \sqrt{\int_{-\infty}^{\infty} f^2(x)dx}$ was:

$$\frac{\|G_{60}(x,3) - G_{100}(x,3)\|}{\|G_{60}(x,3)\|}100\% = 3.2\% \tag{36}$$

where $G_{60}(x,3)$ – influence function considering the first sixty members, and $G_{100}(x,3)$ – influence function considering the first one hundred members in series (33).

In Figure 3 a comparison of the results received with the help of the first and second offered methods for inverse of Laplace and Fourier integral transforms.

The solid curve corresponds to the original built following the formula (14) considering the first ten members in series, and dashed curve corresponds to the original built following the formula (33) considering the first sixty members of the correspondent series.

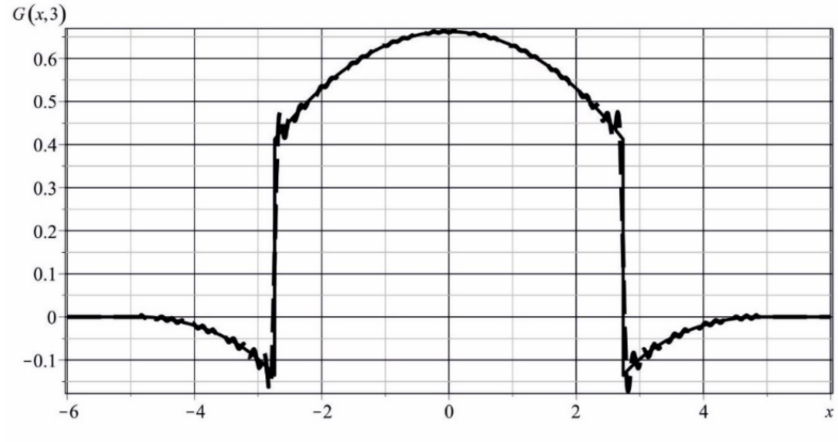

Fig. 3 – Comparison of results received with the help of the first and second offered methods of inverse

It can be seen that the general type of curves is the same, besides there is also a difference in the presence of Gibbs effects near discontinuities at the front of the bending wave as a result received with the help of the second approach to Fourier and Laplace images. So, the first approach to the formation of the original allows emphasizing the breakage of the first kind at the front of bending wave.

The dependencies of the influence functions from time at points with different values coordinates x are represented in Figure 4. The solid curve correspond to the original of the influence function built on formula (14) considering the first ten members of the correspondent series, and dashed lines correspond to the original built on formula (33) considering the first sixty members of the correspondent kind.

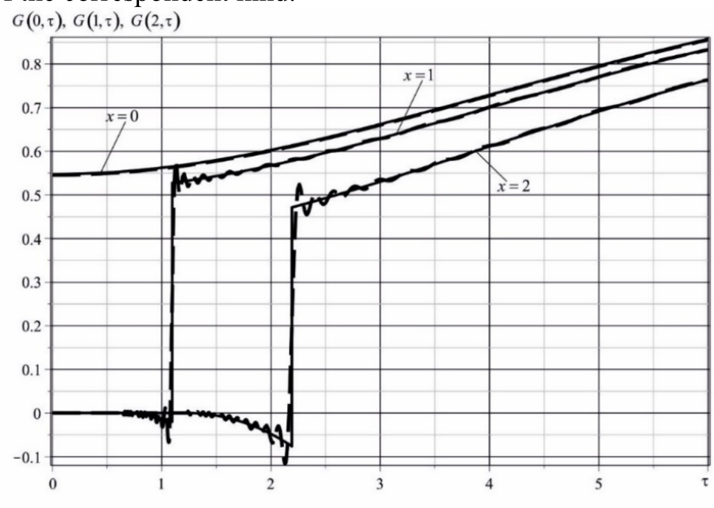

Fig. 4 – Dependencies of influence functions from time

We may study non-stationary vibrations of the beam (or plate in simple case) under the influence of distributed load:

$$p = cos(x)\, e^{-\tau} H(\tau) H(1 - |x|), \tag{37}$$

Graphs can be seen at different time in Figure 5:

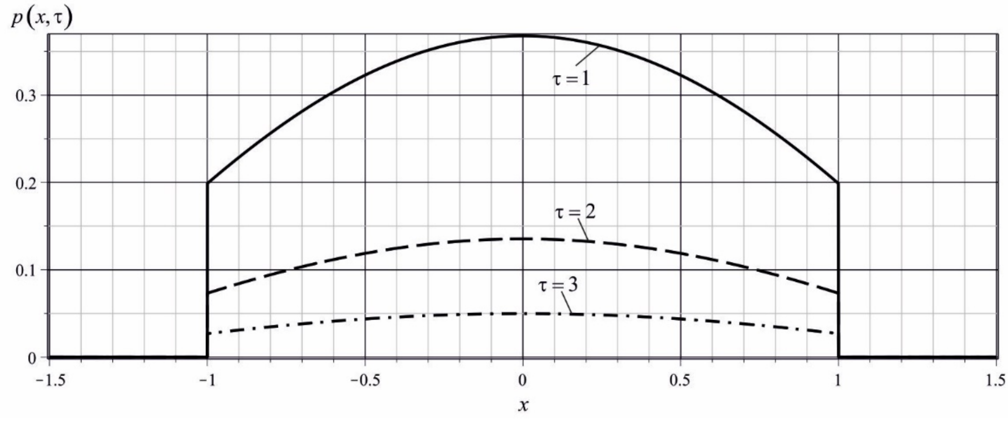

Fig. 5 – Distributed load

The calculation of normal movements is carried out on formula (7). The Simpson method is used for calculating the double integral.

In Figure 6 graphs of beam bending are shown in different time. Besides, in formula (7) the influence function is used built with the help of the first (solid curves) and the second (dashed curves) method.

In the first case the first five members (14) in series are kept and in the second case the first ten members are kept in series (33). It can be seen that the results are practically coincide.

An important conclusion follows from it: the convergence of the results for non-stationary vibrations from actual distributed loads with the influence functions as series (13) and (33) is significantly improved.

This explains by that the influence functions are essentially generalized by functions and the convergence of the corresponding series should be estimated from the position of convergence as a rule [54].

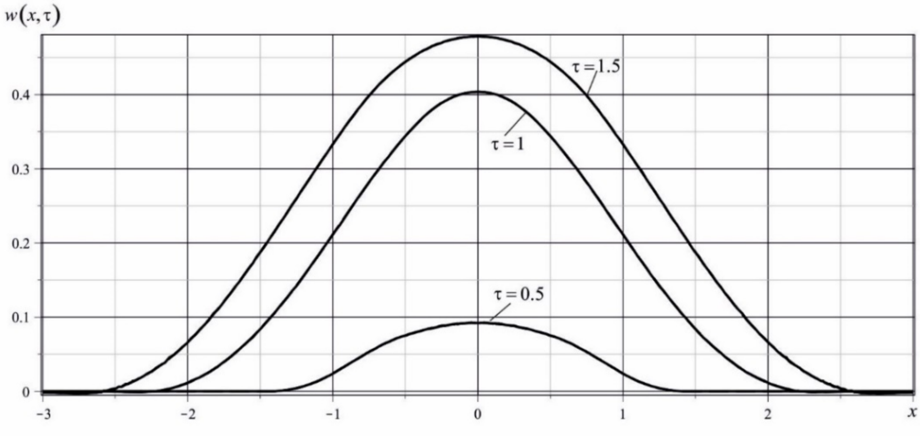

Fig. 6 – Bending $w(x, \tau)$ at different time moments

Dependencies of the deflection from time at point with coordinate $x = 0$ are represented in Figure 7.

The solid curve is still the first five members in series in formula (14), and dashed line is corresponded to the first ten members in series in formula (33).

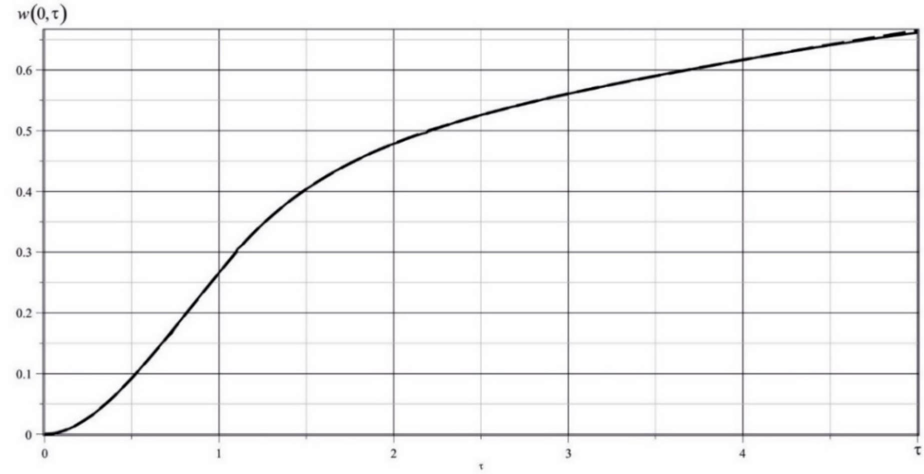

Fig. 7 – Bending depending from time

4. CONCLUSIONS

The problem concerning non-stationary vibrations of Timoshenko beam and plate (in case of simple statement of problem) is solved. The dependence function is built for Timoshenko beam and shown that it coincides with the function of influence for Timoshenko plate (in case of simple statement of the problem). Two methods of reverse of Fourier and Laplace integral transforms are proposed. The features of wave processes in Timoshenko-type beams and plates are discovered. Examples of calculations are given. Results convergence estimate is carried out.

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