Method for solving plane unsteady contact problems for rigid stamp and elastic half-space with a cavity of arbitrary geometry and location

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Abstract: *In the work, the process of unsteady contact interaction of rigid stamp and elastic half-space having a recessed cavity of arbitrary geometry and location with a smooth boundary was investigated. Three variants of contact conditions are considered: free slip, rigid coupling, and bonded contact. The method for solving the problem is constructed using boundary integral equations. To obtain boundary integral equations, the dynamic reciprocal work theorem is used. The kernels of integral operators are bulk Green functions for the elastic plane. Because of straight-line approximations of the domain boundaries with respect to the spatial variable and straight-line approximations of the boundary values of the desired functions with respect to time, the problem is reduced to solving a system of algebraic equations with respect to the pivotal values of the desired displacements and stresses at each time interval. One of the axes is directed along the regular boundary of half-space, the second - deep into half-space.*

Key Words: *boundary integral equations, Green functions, dynamic reciprocal work theorem, inhomogeneity*

1. INTRODUCTION

Many important practical problems are related to the study of the dynamic contact interaction of bounded bodies with semi-bounded elastic domains of complex structure. These problems are connected, inter alia, with the problems of seismic resistance and vibration protection of structures, the calculation of the level and characteristics of the exposure to buildings and structures of technogenic vibrations propagating in the soil, seismic exploration of minerals, etc.

In the soil mass, inhomogeneities (structural disturbances) are often present, both natural (karst cavities, more rigid inclusions) and artificial (various communications, metro tunnels, buried waste storage facilities, etc.) origin. Therefore, the question of the degree of influence of such inhomogeneities on the wave fields generated in an array with an inhomogeneity is significant. In the tasks of designing earthquake-proof buildings and structures, it is important, with a sufficient degree of accuracy, to determine the parameters of unsteady oscillations of objects located on the earth's surface. It should be noted that the placement of any sensors in the contact zone inevitably gives rise to distortion of the stress-strain state in their local neighborhood, which significantly complicates experimental studies of the contact stress distributions, and makes them practically impossible in most cases. This leads to the need to develop theoretical methods and approaches to solving the class of problems under consideration.

Various aspects of solving stationary and unsteady problems for continuous bodies and shells, including taking into account the influence of temperature, were considered in [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

In the field of mechanics of contact interactions, the least studied are unsteady contact problems. To date, there is only a limited range of works devoted to the study of unsteady contact interaction processes for rigid or deformable bodies with elastic half-space [11], [12], [13], [14], [15].

Unsteady contact problems in which half-space has recessed cavities are even less studied. On the other hand, these problems are extremely important for various branches of the national economy, such as geophysics, seismology, acoustics, vibroseis works, foundation engineering, military industry, etc.

The relevance of research is determined by the possibility of its wide practical application in various fields of mechanics. Dynamic problems for semi-infinite media containing buried cavities are currently poorly understood.

The complexity of their study is because due to the multiplicity of the base, the traditional methods for solving unsteady contact problems for simply connected bases, which are usually based on the reduction of the original problem to functional and integral boundary equations, are not applicable here [16], [17], [18], [19]. In this regard, both research on the class of problems in the new formulation and the development of new numerical and analytical methods for solving them become relevant.

2. RESEARCH METHODOLOGY

Unsteady problems are considered for a homogeneous elastic half-space $y \geq 0$, having a buried cavity bounded by a smooth curve γ. To describe the motion of half-space, we use the Cartesian coordinate system $Oxyz$.

The Ox axis is directed along the regular boundary of the half-space, and the Oy is directed deep into half-space.

We assume that the problem is plane: all the required and given functions depend only on two spatial coordinates x, y and time t.

Moreover, the displacement vector $\boldsymbol{u}$ has two nonzero components: $u(x, y, t)$ – along the Ox axis and $w(x, y, t)$ – along the Oy axis (Fig. 1).

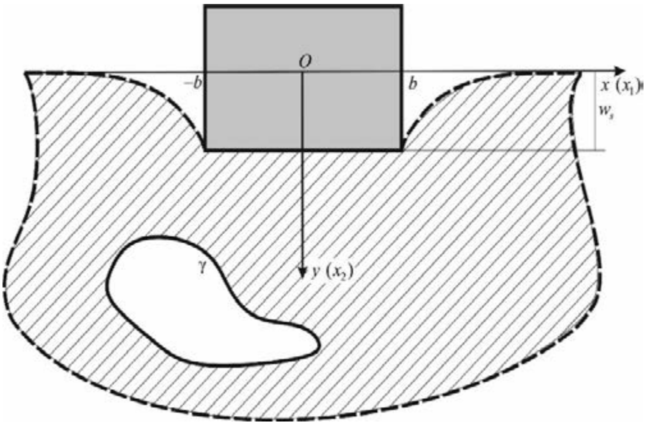

Fig. 1 - Statement of the problem

We introduce a system of dimensionless quantities (the dash mark denotes dimensional parameters)

$$x = \frac{x'}{L} \tag{1}$$

$$y = \frac{y'}{L} \tag{2}$$

$$\tau = \frac{c_1 t}{L} \tag{3}$$

$$u = \frac{u'}{L} \tag{4}$$

$$w = \frac{w'}{L} \tag{5}$$

$$F_i = \frac{F_i' L}{\lambda + 2\mu} \tag{6}$$

$$\eta = \frac{c_1}{c_2} \tag{7}$$

$$c_1^2 = \frac{\lambda + 2\mu}{\rho} \tag{8}$$

$$c_2^2 = \frac{\mu}{\rho} \tag{9}$$

$$\sigma_{ij} = \frac{\sigma_{ij}'}{\lambda + 2\mu} \tag{10}$$

Here L is some characteristic, c_1 and c_2 are the propagation velocities of stress-strain and shear waves; $F_i, i = 1{,}2$ – components of mass forces; λ, μ and ρ are the Lame elastic constants and the density of the medium; τ is the nondimensional time, $\sigma_{ij}, i, j = 1{,}2$ are the components of the stress tensor.

Hereinafter, the “1” index of the quantity corresponds to the x coordinate, and the “2” index to the y coordinate.

Further, all equations and relations will be written in a dimensionless form, taking into account the introduced system of dimensionless quantities (1-10).

The motion of an elastic medium is described by the Navier equations [20]:

$$\ddot{u} = (1-\eta^{-2})\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\right)+\eta^{-2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F_1(x,y,\tau) \tag{11}$$

$$\ddot{w} = (1-\eta^{-2})\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\right)+\eta^{-2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+F_3(x,y,\tau) \tag{12}$$

The dots here and below denote the derivatives with respect to nondimensional time τ. For the system of equations (11, 12), you can use the following index entry:

$$\ddot{u}_k = (1-\eta^{-2})\frac{\partial\theta}{\partial x_k}+\eta^{-2}\Delta u_k+F_k(x,y,\tau), k=1,2 \tag{13}$$

where $u_1 = u, u_2 = w,$

$$\theta = \frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}=\frac{\partial u_k}{\partial x_k} \tag{14}$$

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{15}$$

Δ is the two-dimensional buckling. Hereinafter, the repeating Latin indices summarize from 1 to 2.

We also introduce the *differential operator of elastic equilibrium*

$$L_k(\boldsymbol{u}) = -(1-\eta^{-2})\frac{\partial\theta}{\partial x_k}+\eta^{-2}\Delta u_k \tag{16}$$

Then equation (3) can be written in operator form

$$\ddot{u}_k + L_k(\boldsymbol{u}) = F_k(x,y,\tau) \tag{17}$$

If we introduce the *vector operator of elastic equilibrium*

$$\boldsymbol{L} = [L_1(\boldsymbol{u}), L_2(\boldsymbol{u})] \tag{18}$$

Then system of equations (5) can be written in vector form

$$\ddot{\boldsymbol{u}} + \boldsymbol{L}(\boldsymbol{u}) = \boldsymbol{F} \tag{19}$$

Nonzero components of the strain tensor $\varepsilon_{xx,}\, \varepsilon_{xy}$ and ε_{yy} are associated with displacements by the Cauchy relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\right), \varepsilon_{yy} = \frac{\partial w}{\partial y} \tag{20}$$

Stresses σ_{xx},σ_{xy} and σ_{yy} are associated with strains by Hooke's law

$$\sigma_{xx} = \varepsilon_{xx}+(1-2\eta^{-2})\varepsilon_{yy}, \sigma_{xy} = 2\eta^{-2}\varepsilon_{xy}, \sigma_{yy} = \varepsilon_{yy}+(1-2\eta^{-2})\varepsilon_{xx} \tag{21}$$

Hooke's law can be written in index form

$$\sigma_{ij} = (1-2\eta^{-2})\theta\delta_{ij}+2\eta^{-2}\varepsilon_{ij} \tag{22}$$

We assume that at the initial instant of time $\tau = 0$ the half-space is in the unperturbed state, which corresponds to zero initial conditions

$$u(x,z,0) = \dot{u}(x,z,0) = w(x,z,0) = \dot{w}(x,z,0) = 0 \tag{23}$$

On the boundary of the half-space $y = 0$, the following types of boundary conditions can be specified.

1. Displacement boundary conditions:

$$u(x,0,\tau) = U(x,\tau), w(x,0,\tau) = W(x,\tau) \tag{24}$$

where $U(x,\tau)$ and $W(x,\tau)$ are given functions characterizing the tangent and normal displacements of the half-space boundary.

2. Stress boundary conditions:

$$\sigma_{xy}(x,0,\tau) = p_1(x,\tau), \sigma_{yy}(x,0,\tau) = p_2(x,\tau) \tag{25}$$

where $p_1(x,\tau)$ and $p_2(x,\tau)$ are given functions characterizing the tangent and normal stresses at the half-space boundary.

3. Mixed boundary conditions:

$$u(x,0,\tau) = U(x,\tau), w(x,0,\tau) = W(x,\tau), x \in \Gamma_u \tag{26}$$

$$\sigma_{xy}(x,0,\tau) = p_1(x,\tau), \sigma_{yy}(x,0,\tau) = p_2(x,\tau), x \in \Gamma_\sigma \tag{27}$$

where Γ_u is the part of the boundary $y = 0$, at which displacements are specified, and Γ_σ is the part of the boundary $y = 0$, at which stresses are set.

At infinity, displacements are assumed to be limited.

$$u|_{r\to\infty} = O(1),\ w|_{r\to\infty} = O(1),\ r = \sqrt{x^2 + y^2} \tag{28}$$

At the boundary of the cavity γ, it is also possible to specify one of three types of boundary conditions.

1. Displacement boundary conditions:

$$u_s|_\gamma = U_s(s,\tau), u_n|_\gamma = U_n(s,\tau) \tag{29}$$

where $u_s = (\boldsymbol{u},\boldsymbol{s}) = u_k \nu_k$ and $u_n = (\boldsymbol{u},\boldsymbol{n}) = u_k n_k$ are the projections of the displacement vector on the direction of the tangent and normal to the contour γ, $\boldsymbol{n}$, $\boldsymbol{s}$ are the unit outer normal vectors and unit tangent vectors to the contour γ. $U_s(s,\tau)$, $U_n(s,\tau)$ are given functions of the arc length of the curve γ and time characterizing the tangent and normal displacements at the boundary of the cavity.

2. Stress boundary conditions:

$$\sigma_s|_\gamma = p_s(s,\tau), \sigma_n|_\gamma = p_n(s,\tau) \tag{30}$$

where σ_ν and σ_n are the tangent and normal stresses on the contour γ; $p_s(s,\tau)$ and $p_n(s,\tau)$ are functions of the arc length of the curve γ and time, which characterize the tangent and normal load at the boundary of the cavity.

3. Mixed boundary conditions:

$$u_s|_{\gamma_u} = U_s(s,\tau), u_n|_{\gamma_u} = U_n(s,\tau) \tag{31}$$

$$\sigma_s|_{\gamma_\sigma} = p_s(s,\tau), \sigma_n|_{\gamma_\sigma} = p_n(s,\tau) \tag{32}$$

where γ_u is the part of the cavity boundary on which displacements are specified, and γ_σ is the part of the cavity boundary on which loads are specified.

In the case of contact interaction of the half-space boundary $\boldsymbol{y=0}$ with a rigid stamp with a half-width $\boldsymbol{b}$ (Fig. 1), in the contact region $\boldsymbol{x} \in [-\boldsymbol{b}, \boldsymbol{b}]$, the following contact conditions can be realized ($\boldsymbol{w_s = w_s(\tau)}$ – stamp drift, which depends on time).

1. Free slip conditions.

$$w|_{y=0} = w_s, \sigma_{xy}\big|_{y=0} = 0, \sigma_{yy}\big|_{y=0} < 0, x \in [-b, b] \tag{33}$$

2. Rigid coupling conditions.

$$w|_{y=0} = w_s, u|_{y=0} = 0, \sigma_{yy}\big|_{y=0} < 0, x \in [-b, b] \tag{34}$$

3. Bonded contact.

$$w|_{y=0} = w_s, \sigma_{xy}\big|_{y=0} = k_T \sigma_{yy}\big|_{y=0}, \sigma_{yy}\big|_{y=0} < 0, x \in [-b, b] \tag{35}$$

where k_T is the coefficient of friction. The law of embedding the stamp $w_s(\tau)$ is assumed to be known.

3. RESULTS AND DISCUSSIONS

3.1 Green functions for an elastic plane

To solve the problems posed, we need the Green functions for the elastic plane Oyz. These functions are displacements $G^u_{km}(x,z)$ and stresses $G^\sigma_{klm}(x,z)$ as solutions of problem (5) - (11) bounded at infinity for an infinite elastic plane under the action of unit concentrated mass forces applied at the origin:

$$\ddot{G}^u_{km} - L(G^u_{km}) = \delta_{km}\delta(\tau)\delta(x_1, x_2) \tag{36}$$

$$\theta_m = \frac{\partial G^u_{1m}}{\partial x_1} + \frac{\partial G^u_{2m}}{\partial x_2} = \frac{\partial G^u_{km}}{\partial x_k} = G^\varepsilon_{11m} + G^\varepsilon_{22m} = G^\varepsilon_{kkm} \tag{37}$$

$$G^\varepsilon_{klm} = \frac{1}{2}\left(\frac{\partial G^u_{km}}{\partial x_l} + \frac{\partial G^u_{lm}}{\partial x_k}\right) \tag{38}$$

$$G^\sigma_{klm} = (1 - \eta^{-2})\theta_m \delta_{kl} + 2\eta^{-2} G^\varepsilon_{klm} \tag{39}$$

where δ_{km} is the Kronecker symbol, $\delta(\tau)$, $\delta(x_1, x_2)$ are the Dirac delta functions [16].

Applying to (39) the direct two-dimensional integral Fourier transformation in the spatial coordinates x and y and the Laplace integral transformation in time, and then sequentially inverting the Fourier and Laplace integral transformations using the tables [21], [22], [23] we find the inverse Green functions:

$$\begin{aligned} G^u_{km}(x,y,\tau) = \frac{\delta_{km}}{2\pi r^2}\left[\tau^2(\tau^2 - \eta^2 r^2)_+^{\frac{-1}{2}} - (\tau^2 - r^2)_+^{\frac{1}{2}}\right] \\ - \frac{x_k x_m}{2\pi r^4}\sum_{j=1}^{2}(-1)^j\left(2\tau^2 - \eta_j^2 r^2\right)\left(\tau^2 - \eta_j^2 r^2\right)_+^{\frac{-1}{2}} \end{aligned} \tag{40}$$

$$G_{klm}^{\varepsilon}(x,y,\tau)=\frac{\eta^4}{4\pi}(x_k\delta_{lm}+x_l\delta_{km})(\tau^2-\eta^2r^2)_+^{\frac{-3}{2}}$$
$$+\frac{1}{2\pi r^4}\sum_{j=1}^{2}(-1)^j\left[\frac{x_kx_lx_m}{r^2}\left(8\tau^4-12\eta_j^2r^2\tau^2\right.\right.$$
$$\left.+3\eta^4r^4\right)\left(\tau^2-\eta_j^2r^2\right)_+^{\frac{-3}{2}} \tag{41}$$
$$\left.--(x_k\delta_{lm}+x_l\delta_{km}+x_m\delta_{kl})\left(2\tau^2-\eta_j^2r^2\right)\left(\tau^2-\eta_j^2r^2\right)_+^{\frac{-1}{2}}\right]$$

$$G_{klm}^{\sigma}=(1-\eta^{-2})\theta_m\delta_{kl}+2\eta^{-2}G_{klm}^{\varepsilon} \tag{42}$$

Hereinafter:

$$f(x)_+=\begin{cases}f(x), x\geq 0;\\ 0, x<0.\end{cases} \tag{43}$$

Obviously, the dominant functions G_{km}^{u} and G_{klm}^{ε} (therefore, G_{klm}^{σ}) are symmetric in the indices k,m and k,l: $G_{km}^{u}=G_{mk}^{u}, G_{klm}^{\sigma}=G_{lkm}^{\sigma}$

3.2 An analysis of the reciprocal work theorem of the two-dimensional unsteady theory of elasticity and the main resolving boundary integral equation

Let's consider a certain two-dimensional domain D, finite (bounded by the curve Γ) or infinite. Domain D can also be semirestricted, for example, be a half-plane. Consider in the domain D two displacement fields defined by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. For them, the dynamic reciprocal work theorem holds [8]

$$\iint_D \boldsymbol{F}(\boldsymbol{u})*\boldsymbol{v}d\Omega+\int_\Gamma \boldsymbol{p}(\boldsymbol{u})*\boldsymbol{v}ds=\iint_D \boldsymbol{F}(\boldsymbol{v})*\boldsymbol{u}d\Omega+\int_\Gamma \boldsymbol{p}(\boldsymbol{v})*\boldsymbol{u}ds \tag{44}$$

where $\boldsymbol{p}$ is the stress vector on the contour Γ, the symbol "*" means the convolution operation in time ($\boldsymbol{x}=(x_1,x_2)$):

$$\boldsymbol{F}(\boldsymbol{u})*\boldsymbol{v}=\int_0^\tau \boldsymbol{F}[\boldsymbol{u}(\boldsymbol{x},\tau-t)]\boldsymbol{v}(\boldsymbol{x},t)dt \tag{45}$$

In the absence of mass forces $\boldsymbol{F}(\boldsymbol{u})\equiv 0$ in formula (44), two-dimensional integrals will be equal to zero. Further, we will assume that there are no mass forces. Thus, having some trial state $\boldsymbol{v}$, $\boldsymbol{p}(\boldsymbol{v})$, for the desired solution $\boldsymbol{u}$, $\boldsymbol{p}(\boldsymbol{u})$ we obtain the boundary integral equation of the plane unsteady problem

$$\int_\Gamma \boldsymbol{v}_1*\boldsymbol{p}(\boldsymbol{u})ds=\int_\Gamma \boldsymbol{p}(\boldsymbol{v}_1)*\boldsymbol{u}ds \tag{46}$$

Equation (27) is conveniently represented in the component notation. As follows from the formulations of the initial boundary value problems, in the boundary conditions that are set on the contour Γ, the normal and tangent components of the displacement vectors $U_n(s,\tau)$, $U_s(s,\tau)$ and forces $p_n(s,\tau)$, $p_s(s,\tau)$ are presented. Therefore, it is convenient to take the tangent and normal to the displacement contour u_s, u_n and the tangent and normal stresses σ_s, σ_n as the desired displacements and stresses. Similarly, other normal to the displacement contour u_s', u_n' are used as the trial state. Then, obviously, equation (27) takes the form

$$\int_{\Gamma} \sigma_s * u_s' + \sigma_n * u_n' ds = \int_{\Gamma} \sigma_s' * u_s + \sigma_n' * u_n ds \tag{47}$$

Equation (47) is the main one and will be used to solve initial boundary value problems.

3.3 Features of the discrete analog of the boundary integral equation

Let's consider a certain domain D, bounded by the contour Γ and filled with an elastic medium. With each point $\boldsymbol{\xi}$ of the contour Γ we associate the tangent and normal displacements u_s, u_n and the tangent and normal stresses (or forces) σ_s, σ_n. These quantities are specified relative to the local coordinate system s, n of the point $\boldsymbol{\xi}$ (Fig. 2).

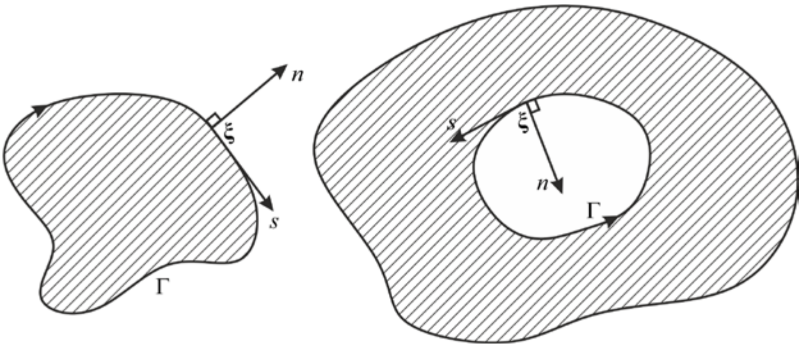

Fig. 2 - Local coordinate systems and directions of traversal of the contour Γ in the case of internal and external domains with respect to Γ.

Moreover, at each point of the contour, the tangent stress σ_s or the tangent displacement u_s and the normal stress σ_n or the normal displacement u_n are specified, i.e., two of the four quantities u_s, u_n, σ_s and σ_n are known in advance from the boundary conditions. The remaining two quantities should be found from the solution of the problem. For this, we use equation (28). We choose some trial state characterized by displacements u_s', u_n' and stresses σ_s', σ_n'.

To solve equation (47) numerically, we apply time sampling. To do this, divide the time interval $[0, \tau]$ into N, equal intervals of duration:

$$\Delta t: t_k = k\Delta t \tag{48}$$

$$k = 1,2,\ldots,N \tag{49}$$

$$\tau = N\Delta t \tag{50}$$

The displacements and stresses in the initial problem are approximated linearly in time:

$$u_n(s,t) = u_n^k(s)m_1(t) + u_n^{k-1}(s)m_2(t), u_s = u_s^k(s)m_1(t) + u_s^{k-1}(s)m_2(t) \tag{51}$$

$$\sigma_n = \sigma_n^k(s)m_1(t) + \sigma_n^{k-1}(s)m_2(t) \tag{52}$$

$$\sigma_s = \sigma_s^k(s)m_1(t) + \sigma_s^{k-1}(s)m_2(t) \tag{53}$$

$$u_n^k(s) = u_n(s,t_k) \tag{54}$$

$$u_s^k(s) = u_s(s,t_k) \tag{55}$$

$$\sigma_n^k(s) = \sigma_n(s,t_k) \tag{56}$$

$$\sigma_s^k(s) = \sigma_s(s, t_k) \tag{57}$$

Substitution of (29) into (28) brings the latter to the form:

$$\begin{aligned} &\sum_{k=1}^{N} \int_{\Gamma} \sigma_s^k u'^k_{s1} + \sigma_s^{k-1} u'^k_{s2} ds + \sigma_n^k u'^k_{n1} + \sigma_n^{k-1} u'^k_{n2} ds \\ &\qquad = \sum_{k=1}^{N} \int_{\Gamma} u_s^k \sigma'^k_{s1} + u_s^{k-1} \sigma'^k_{s2} + u_n^k \sigma'^k_{n1} + u_n^{k-1} \sigma'^k_{n2} ds \end{aligned} \tag{58}$$

We approximate the contour Γ using M rectilinear segments adjoining each other (Fig. 3):

$$\Gamma \approx \bigcup_{j=1}^{M} \gamma_j \tag{59}$$

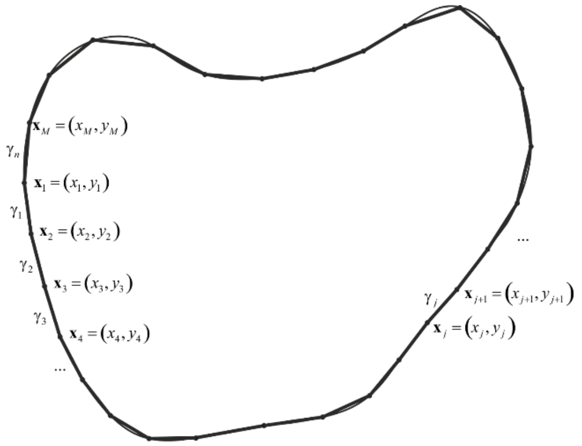

Fig. 3 - Approximation of the contour Γ

Now we assume that the displacements and stresses at the boundary of the contour Γ within each segment γ_j are constant, and then equation (30) takes the form:

$$\begin{aligned} &\sum_{k=1}^{N} \sum_{j=1}^{M} \sigma_s^{k,j} \int_{\gamma_j} u'^k_{s1} ds + \sigma_s^{k-1,j} \int_{\gamma_j} u'^k_{s2} ds + \sigma_n^{k,j} \int_{\gamma_j} u'^k_{n1} ds + \sigma_n^{k-1,j} \int_{\gamma_j} u'^k_{n2} ds \\ &\qquad = \sum_{k=1}^{N} \sum_{j=1}^{M} u_s^{k,j} \int_{\gamma_j} \sigma'^k_{s1} ds + u_s^{k-1,j} \int_{\gamma_j} \sigma'^k_{s2} ds + u_n^{k,j} \int_{\gamma_j} \sigma'^k_{n1} ds \\ &\qquad + u_n^{k-1,j} \int_{\gamma_j} \sigma'^k_{n2} ds \end{aligned} \tag{60}$$

By rearranging the terms taking into account zero initial conditions, equation (60) can be rewritten as follows:

$$\sum_{j=1}^{M} \left(u_s^{N,j} \sigma'^N_{s,j} + u_n^{N,j} \sigma'^N_{n,j} - \sigma_s^{N,j} u'^N_{s,j} - \sigma_n^{N,j} u'^N_{n,j} \right) = F_N \tag{61}$$

$$\sigma'^{N}_{s,j} = \int_{\gamma_j} \sigma'^{N}_{s1} ds \,, \sigma'^{N}_{n,j} = \int_{\gamma_j} \sigma'^{N}_{n1} ds \,, u'^{N}_{s,j} = \int_{\gamma_j} u'^{N}_{s1} ds \,, u'^{N}_{n,j} = \int_{\gamma_j} u'^{N}_{n1} ds \tag{62}$$

$$\begin{aligned} F_N = &\sum_{k=1}^{N-1} \sum_{j=1}^{M} \sigma_s^{k,j} \int_{\gamma_j} \left(u'^{k}_{s1} + u'^{k+1}_{s2}\right) ds + \sigma_n^{k,j} \int_{\gamma_j} \left(u'^{k}_{n1} + u'^{k+1}_{n2}\right) ds \\ &- \sum_{k=1}^{N-1} \sum_{j=1}^{M} u_s^{k,j} \int_{\gamma_j} \left(\sigma'^{k}_{s1} + \sigma'^{k+1}_{s2}\right) ds + u_n^{k,j} \int_{\gamma_j} \left(\sigma'^{k}_{n1} + \sigma'^{k+1}_{n2}\right) ds \end{aligned} \tag{63}$$

In equations (61-63), the right-hand side F_N is known, because it contains the required functions at the preceding current time steps ($k = 1,2,\dots,N-1$). In accordance with the given boundary conditions, the left-hand side of equations (61-63) contains $2M$ unknown pivotal values of displacements or stresses.

The remaining $2M$ pivotal values are given by the boundary conditions (29), (30) or (31), (32). In addition, in the case of contact problems, the corresponding contact conditions (33), (34), or (21) are specified on a part of the boundary Γ.

Therefore, the $2M$ terms on the left-hand side of equations (61-63) are also known in each case. We also note that in order to obtain a closed system of resolving equations for $2M$ of the required pivotal values of displacements or stresses, it is necessary to form $2M$ equations of the form (61-63). For this, it is necessary to provide a sufficient number, namely, $2M$ trial solutions (states) u'_s, u'_n, σ'_s, σ'_n.

3.4 The choice of trial solutions and the formation of a closed system of resolving equations

As indicated in the previous paragraph, for the formation of a closed system of resolving equations, it is necessary to have $2M$ trial solutions. We assume that in the unbounded elastic plane filled with the elastic medium there is a “fictitious” contour $\tilde{\Gamma}$, the position of which coincides with the position of the contour Γ.

The contour $\tilde{\Gamma}$ is approximately replaced by a piecewise linear approximation, as indicated in Fig. 3:

$$\tilde{\Gamma} \approx \bigcup_{j=1}^{M} \gamma_j \tag{64}$$

Further, since in the domain $\widetilde{D}$, bounded by the contour $\tilde{\Gamma}$ and the corresponding domain D, the constructed solution will coincide with the required one, we will not distinguish between the contour Γ and the fictitious contour $\tilde{\Gamma}$, as well as between the domains D and $\widetilde{D}$.

As trial solutions, we take solutions to the problems of the action of normal P_s^i concentrated along coordinates and time and unit forces tangent P_n^i, applied to the points ξ_i, which are the midpoints of the segments γ_i, $i = 1,2,\dots,M$ *from outside* the domain D, bounded by the contour Γ.

The solutions to these problems are the dominant functions (see § 1.4).

To calculate the coefficients and the right-hand side of equations (61-63), it is convenient to introduce a local coordinate system with $\tilde{x}$ (s_j) and $\tilde{y}$ (n_j) axes on each γ_j segment, and the $\tilde{y}$ (n_j) coincides with the direction of the external normal to the contour Γ, and the axis $\tilde{y}$ (n_j) is directed in the direction of the contour Γ (Fig. 4).

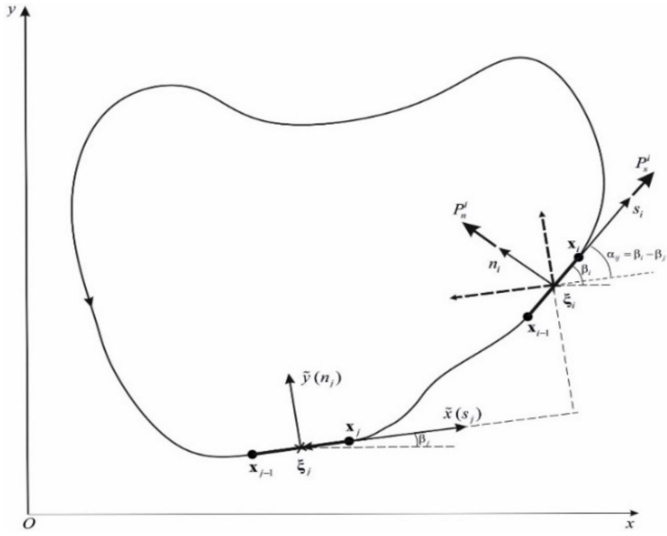

Fig. 4 - Trial states

Let the unit normal force P_n^i and the unit tangent force P_s^i be applied at the point $\boldsymbol{\xi}_i$ of the segment γ_i. Then, in the local coordinate system $\tilde{x}$ (s_j) and $\tilde{y}$ (n_j) the total projections of these forces on the coordinate axes will be determined by the following expressions:

$$P_{\tilde{x}}^{ij} = P_s^i \cos\alpha_{ij} - P_n^i \sin\alpha_{ij,} P_{\tilde{y}}^{ij} = P_s^i \sin\alpha_{ij} + P_n^i \cos\alpha_{ij} \tag{65}$$

where

$$\alpha_{ij} = \beta_i - \beta_j \tag{66}$$

β_i and β_j are the angles between the vectors $\boldsymbol{s}_i$, $\boldsymbol{s}_j$ with the Ox axis of the global Cartesian coordinate system Oxy. Moreover, trial solutions in the local coordinate system, according to paragraph 2, have the form:

$$\begin{aligned}\sigma'_{\text{xy}} = \sigma'_s = {} & P_s^i G_{121}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij} - P_n^i G_{121}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} \\ & + P_s^i G_{122}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} \\ & + P_n^i G_{122}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij}\end{aligned} \tag{67}$$

$$\begin{aligned}\sigma'_{\text{yy}} = \sigma'_n = {} & P_s^i G_{221}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij} - P_n^i G_{221}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} \\ & + P_s^i G_{222}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} \\ & + P_n^i G_{222}^{\sigma}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij}\end{aligned} \tag{68}$$

$$\begin{aligned}u'_{\tilde{x}} = u'_s = {} & P_s^i G_{11}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij} - P_n^i G_{11}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} \\ & + P_s^i G_{12}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} + P_n^i G_{12}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij}\end{aligned} \tag{69}$$

$$\begin{aligned}u'_{\tilde{y}} = u'_n = {} & P_s^i G_{21}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij} - P_n^i G_{21}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} \\ & + P_s^i G_{22}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \sin\alpha_{ij} + P_n^i G_{22}^{u}(\tilde{x} - c_{ij}, -d_{ij}, \tau) \cos\alpha_{ij}\end{aligned} \tag{70}$$

Note that formulas (67-70) essentially contain two trial solutions: one due to the action of the unit force P_s^i, and the other due to the action of the unit force P_n^i. Assuming in (67-70) $P_s^i = 1$, $P_n^i = 0$, we obtain the first trial solution for the element γ_i:

$$\sigma'_{s1,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = G^{\sigma}_{121}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} + G^{\sigma}_{122}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} \quad (71)$$

$$\sigma'_{n1,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = G^{\sigma}_{221}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} + G^{\sigma}_{222}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} \quad (72)$$

$$u'_{s1,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = G^{u}_{11}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} + G^{u}_{12}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} \quad (73)$$

$$u'_{n1,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = G^{u}_{21}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} + G^{u}_{22}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} \quad (74)$$

Similarly, assuming in (67-70) $P_s^i = 0$, $P_n^i = 1$, we obtain the second trial solution for the element γ_i:

$$\sigma'_{s2,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = -G^{\sigma}_{121}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} + G^{\sigma}_{122}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} \quad (75)$$

$$\sigma'_{n2,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = -G^{\sigma}_{221}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} + G^{\sigma}_{222}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij}, \quad (76)$$

$$u'_{s2,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = -G^{u}_{11}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} + G^{u}_{12}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} \quad (77)$$

$$u'_{n2,ij}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right) = -G^{u}_{21}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\sin\alpha_{ij} + G^{u}_{22}\left(\tilde{x}-c_{ij},-d_{ij},\tau\right)\cos\alpha_{ij} \quad (78)$$

Since the number of elements γ_i is equal to M, repeating the same steps for each element, we get $2M$ of necessary trial solutions. For each pair of these solutions from (61-63) we obtain two equations:

$$\sum_{j=1}^{M}\left(u_s^{N,j} a_{ss,ij}^N + u_n^{N,j} a_{ns,ij}^N - \sigma_s^{N,j} b_{ss,ij}^N - \sigma_n^{N,j} b_{ns,ij}^N\right) = F_{si}^N \quad (79)$$

$$F_{si}^N = \sum_{k=1}^{N-1}\sum_{j=1}^{M} \sigma_s^{k,j} b_{ss,ij}^k + \sigma_n^{k,j} b_{ns,ij}^k - u_s^{k,j} a_{ss,ij}^k - u_n^{k,j} a_{ns,ij}^k \quad (80)$$

$$\sum_{j=1}^{M}\left(u_s^{N,j} a_{ss,ij}^N + u_n^{N,j} a_{ns,ij}^N - \sigma_s^{N,j} b_{ss,ij}^N - \sigma_n^{N,j} b_{ns,ij}^N\right) = F_{ni}^N \quad (81)$$

$$F_{ni}^N = \sum_{k=1}^{N-1}\sum_{j=1}^{M} \sigma_s^{k,j} b_{sn}^k + \sigma_n^{k,j} b_{nn,ij}^k - u_s^{k,j} a_{sn,ij}^k - u_n^{k,j} a_{nn,ij}^k \quad (82)$$

$$a_{ss,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{ss,ij}^{N}dx\,,a_{ns,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{ns,ij}^{N}dx\,,b_{ss,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}U_{ss,ij}^{N}dx\,,b_{ns,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{ns,ij}^{N}dx \tag{83}$$

$$b_{ss,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}U_{ss,ij}^{k}dx\,,b_{ns,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}U_{ns,ij}^{k}dx\,,a_{ss,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{ss,ij}^{k}dx\,,a_{ns,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{ns,ij}^{k}dx \tag{84}$$

$$a_{sn,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{sn,ij}^{N}dx\,,a_{nn,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{nn,ij}^{N}dx\,,b_{ss,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}U_{sn,ij}^{N}dx\,,b_{nn,ij}^{N}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{nn,ij}^{N}dx \tag{85}$$

$$b_{sn,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}U_{sn,ij}^{k}dx\,,b_{nn,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}U_{nn,ij}^{k}dx\,,a_{sn,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{sn,ij}^{k}dx\,,a_{nn,ij}^{k}=\int_{-0.5\Delta_j}^{0.5\Delta_j}\Sigma_{nn,ij}^{k}dx \tag{86}$$

$$U_{sn,ij}^{k}=\int_{\tau_{k-1}}^{\tau_k}u'_{s2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_1(t)\,dt+\int_{\tau_k}^{\tau_{k+1}}u'_{s2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_2(t)\,dt \tag{87}$$

$$U_{nn,ij}^{k}=\int_{\tau_{k-1}}^{\tau_k}u'_{n2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_1(t)\,dt+\int_{\tau_k}^{\tau_{k+1}}u'_{n2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_2(t)\,dt \tag{88}$$

$$\Sigma_{sn,ij}^{k}=\int_{\tau_{k-1}}^{\tau_k}\sigma'_{s2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_1(t)\,dt+\int_{\tau_k}^{\tau_{k+1}}\sigma'_{s2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_2(t)\,dt \tag{89}$$

$$\Sigma_{nn,ij}^{k}=\int_{\tau_{k-1}}^{\tau_k}\sigma'_{n2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_1(t)\,dt+\int_{\tau_k}^{\tau_{k+1}}\sigma'_{n2,ij}\big(x-c_{ij},-d_{ij},\tau-t\big)m_2(t)\,dt \tag{90}$$

Note that the coefficients of systems of equations (39), (40), which are the corresponding integrals of the dominant functions of the elastic plane, can contain singularities of order $x^{-\alpha}$, $\alpha>0$. In the case when $\alpha\le\frac{1}{2}$, the corresponding feature will be weak, in the sense that the integral of a function with such a feature exists as an improper integral of the second kind.

In the case when $\alpha > \frac{1}{2}$, the singularity is strong, and the corresponding integral is singular and is understood in the sense of the main value:

$$\int_{-a}^{a} \frac{f(x)}{x^{\alpha}} = \int_{-a}^{a} \frac{f(x) - f(0) - \sum_{m=1}^{[\alpha]} \frac{f^{(m)}(0)}{m!} x^{m}}{x^{\alpha}} dx + \sum_{m=1}^{[\alpha]} \frac{f^{(m)}(0)}{m!} \int_{-a}^{a} \frac{dx}{x^{\alpha-m}} \tag{91}$$

$$\int_{-a}^{a} \frac{dx}{x^{\alpha-m}} = -\frac{a^{m-\alpha-1}}{\alpha-m+1} [1 + (-1)^{m-\alpha-1}] \tag{92}$$

where $[\alpha]$ means the integer part of the number α.

4. CONCLUSIONS

A statement is given and a method for solving new plane unsteady contact problems for rigid stamps and an elastic half-space containing a buried cavity with a smooth boundary of arbitrary geometry is developed. The motion of the elastic half-space is described by the Navier equations in displacements. The statement of the problem also includes Cauchy relations and Hooke's law. At the initial time, the half-space with the cavity is at rest, which leads to zero initial conditions. Outside the contact zone, the surface of the half-space is assumed to be free of stresses, and in the contact domain, conditions of free slip, rigid coupling, or bonded contact can be specified. The solution method is based on the dynamic reciprocal work theorem. Application of the reciprocal work theorem leads to two-dimensional boundary integral equations whose kernels are dominant functions. To solve the resulting system of equations, the direct method of boundary elements with time sampling is used.

As fundamental solutions, the dominant functions for the elastic space are used taking into account the plane formulation of the problem. They determine displacements and stresses in the elastic plane from the applied unit instantaneous concentrated force and act as kernels that resolve boundary integral equations. The integral operators of the resolving system of equations are replaced by discrete analogs in the spatial variable and in time. As a result, at each time interval, the problem reduces to solving a system of algebraic equations. The developed method and solution algorithm allow us to study the processes of unsteady contact interaction of rigid bodies with an elastic half-space having buried cavities of arbitrary geometry and location.

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