Linear Quadratic Gaussian Regulator for the Nonlinear Observer-Based Control of a Dynamic Base Inverted Pendulum

K. Z. MIRZA*

*Corresponding author
Department of Mechatronics, MPSTME, NMIMS,
Mumbai - 400058, India,
Khojasteh.mirza40@nmims.edu.in

DOI: 10.13111/2066-8201.2021.13.3.7

Received: 12 April 2021/ Accepted: 15 July 2021/ Published: September 2021

Abstract: *The inverted pendulum is a non-linear control problem permanently tending towards instability. The main aim of this study is to design a controller capable enough to work within the given conditions while also keeping the pendulum erect given the impulsive movement of the cart to which it is joint via a hinge. The first half of the paper presents the mathematical modelling of the dynamic system, together with the design of a linear quadratic regulator (LQR). This paper also discusses a novel adaptive control mechanism employing a Kalman filter for the mobile inverted pendulum system (MIPS). In the second half of the paper, a Gaussian Quadratic Linear Controller (LQG) is adapted to improve on previous deficiencies. The simulation is done through Simulink and results show that both controllers are capable of managing the multiple output model. However, data from simulations clearly showed that an LQG controller is a better choice.*

Key Words: *Linear Quadratic Regulator, Kalman Filter, Linear Quadratic Gaussian Controller, Inverted Pendulum*

NOMENCLATURE

(M)	mass of the cart
(m)	mass of the pendulum
(b)	coefficient of friction for cart
(l)	length of pendulum
(I)	mass moment of inertia of the pendulum
(F)	*the* force applied to the cart
(x)	cart position coordinate
(u)	cart velocity
(X_G)	X co-ordinate of Instantaneous centre of gravity of point mass
(X_G)	Y co-ordinate of Instantaneous centre of gravity of point mass
(θ)	pendulum angle from vertical (down)
(Φ)	angle of deviation

1. INTRODUCTION

The mobile inverted pendulum system (MIPS) is a typical nonlinear control problem suitable for studying various aspects of an under actuated system. Most of the realistic control systems are non-linear in nature. Controlling such a dynamic system is a challenging task. The control mechanism adopted here can also be used in various military and robotics, chaotic systems, and aerospace applications such as attitude control of launch vehicles and missiles [1-3]. The implementation of a controller for such a system has been of interest to the control engineering community for quite a while due to its complex nature for creating a benchmark to compare different control methods and various procedures for their design. Due to its importance, this was the choice of dynamic system selected. There are various issues at hand with such a system, the biggest being its inertia. Several methods have been adopted to control such a system. In [1], [4-9] many different kinds of control schemes are presented. Shireen and Patel [6] used a basic LQR + PID controller to control the nonlinear system, a common technique for such dynamic situations. The clear drawback, in this case, was the settling time which appears to be about 5 seconds. Efforts were made in this study to address and provide a solution to this issue. Babushanmugham et. al [7] employed Particle Swarm Optimisation (PSO) with Genetic Algorithm and SMC, but PSO-SMC responses proved to show better results. In [8], a comparison is made between a Fuzzy controller and a PID controller, and a conclusion is drawn that a simple PID controller would fail to provide the required stability. The main aim of this study is to design a controller capable enough to work within the given conditions while also keeping the pendulum erect given the impulsive movement of the cart. The system in question considers a cart capable of bidirectional linear motion along the X-axis with an inverted pendulum attached to it via a hinge allowed to oscillate in the X-Y plane. The design parameters are shown in Table 1:

Table 1: Design parameters of the MIPS

Mass of the Cart	0.5 kg
Mass of the Pendulum	0.2 kg
Length of the Pendulum	0.3 m
Coefficient of static friction between Cart and linear rail	0.1
Pendulum's Moment of Inertia	0.006 kgm^2
Max settling time	3 s
Maximum Angular Deviation of the Pendulum	0.5 rad

2. MODELLING

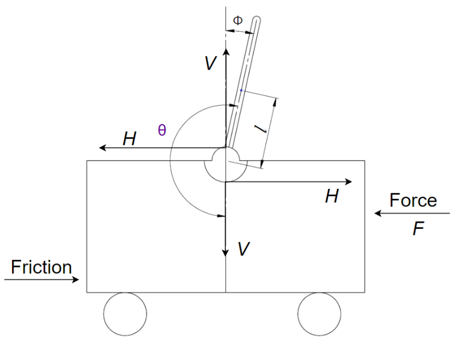

Fig. 1: Free Body Diagram of the System

By summing the forces in the horizontal direction acting on the cart we obtain the first equation of motion.

$$M\ddot{x} + b\dot{x} \tag{1}$$

Summing the forces on the pendulum

$$H = m\dot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta \tag{2}$$

By (2) and (1) the first equation of motion is obtained

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = F \tag{3}$$

By solving about the axis perpendicular to the pendulum

$$Vsin\theta + Hcos\theta - mgsin\theta = ml\ddot{\theta} + m\ddot{x}cos\theta \tag{4}$$

Summing the moments about the centroid of the pendulum,

$$-Vlsin\theta - Hlcos\theta = I\ddot{\theta} \tag{5}$$

From (4) and (5)

$$(I + ml^2)\ddot{\theta} + mglsin\theta = -ml\ddot{x}cos\theta \tag{6}$$

The set of equations of motion (3) and (6) are nonlinear and need to be linearized. It is evident that the output is not proportional to the change in input and for solving such nonlinear equations we employ the standard state-space forms of these two equations.

Linearization of the system about the central upright position of the pendulum is done and a reference state is defined where no external force is applied yet.

Thus, the cart velocity and position are considered to be zero. Here we assume that the system stays within a region cantered at $\theta = \pi$.

On presuming (Φ) a small deviation from the central position, we employ the following approximations:

$$cos\theta = \cos(\pi + \varphi) \approx 1 \tag{7}$$

$$sin\theta = \sin(\pi + \varphi) \approx -\varphi \tag{8}$$

$$\dot{\theta^2} = \dot{\varphi}^2 \approx 0 \tag{9}$$

Substituting (7), (8), and (9) into (3) and (6) we arrive at a linearized form of the equations of motion

$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\varphi} = F \tag{10}$$

$$(I + ml^2)\ddot{\varphi} - mgl\varphi = ml\ddot{x} \tag{11}$$

Taking the Laplace transform of (10) and (11)

$$(M+m)X(s)s^2 + bX(s)s - ml\Phi(s)s^2 = U(s) \tag{12}$$

$$(I + ml^2)\Phi(s)s^2 + bX(s)s - ml\Phi(s) = mlX(s)s^2 \tag{13}$$

We define the output as the angle of deviation (Φ) and the input to the system as U(s).
Re-arranging (13)

$$X(s) = \left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s) \tag{14}$$

Substituting (14) in (12)

$$(M+m)\left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s)s^2 + b\left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s)s - ml\Phi(s)s^2 \tag{15}$$

Thus, we obtain a fourth-order transfer function as:

$$\frac{\Phi(s)}{U(s)} = \frac{\left(\frac{(I+ml^2)s^2 - gml}{q}\right)}{s^4 + \left(\frac{b(I+ml^2)}{q}\right)s^3 - \left(\frac{(M+m)mgl}{q}\right)s^2 - \left(\frac{bmgl}{q}\right)s} \tag{16}$$

where $q = [(M+m)(I+ml^2) - (ml)^2]$

We create a general linear state-space representation for our system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{17}$$

$$y(t) = C(t) + D(t)u(t) \tag{18}$$

The state variables are defined as

$$x_1 = x, x_2 = \dot{x} = \dot{x}_1,\ x_3 = \Phi, x_4 = \dot{\Phi} = \dot{x}_3 \tag{19}$$

For creating the state space representation, we define the most general representation of a linear system and using equations (10) and (11)

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\phi}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ \frac{I+ml^2}{I(M+m)+Mml^2}\\ 0\\ \frac{ml}{I(M+m)+Mml^2}\end{bmatrix} \tag{20}$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u \tag{21}$$

Putting in the aforementioned values, we obtain the matrices as

$$A = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -0.1818 & 2.6727 & 0\\ 0 & 0 & 0 & 1\\ 0 & -0.4545 & 31.1818 & 0\end{bmatrix};\ B = \begin{bmatrix}0\\ 1.8182\\ 0\\ 4.5455\end{bmatrix} \tag{22}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{23}$$

To observe the preliminary stability of our system, a pole-zero plot is created for the previously derived transfer function.

Poles are found at 0, -0.143, -5.6 and 5.57. It is observed that one of the poles lies to the right-hand side of the Y-axis.

Thus, we can assume that such an independent open loop system without a controller would prove to be unstable.

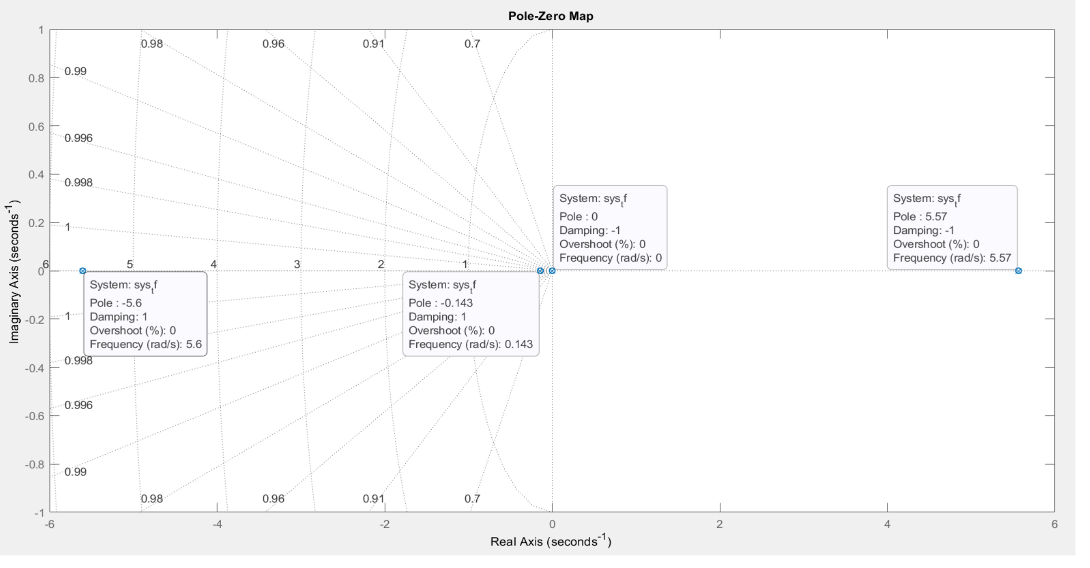

Fig. 2: Pole-Zero plot

By providing an impulse and a step response to our open system, we can justify the above assumption, see Fig. 3. It is observed that the amplitude of the open system is increasing exponentially.

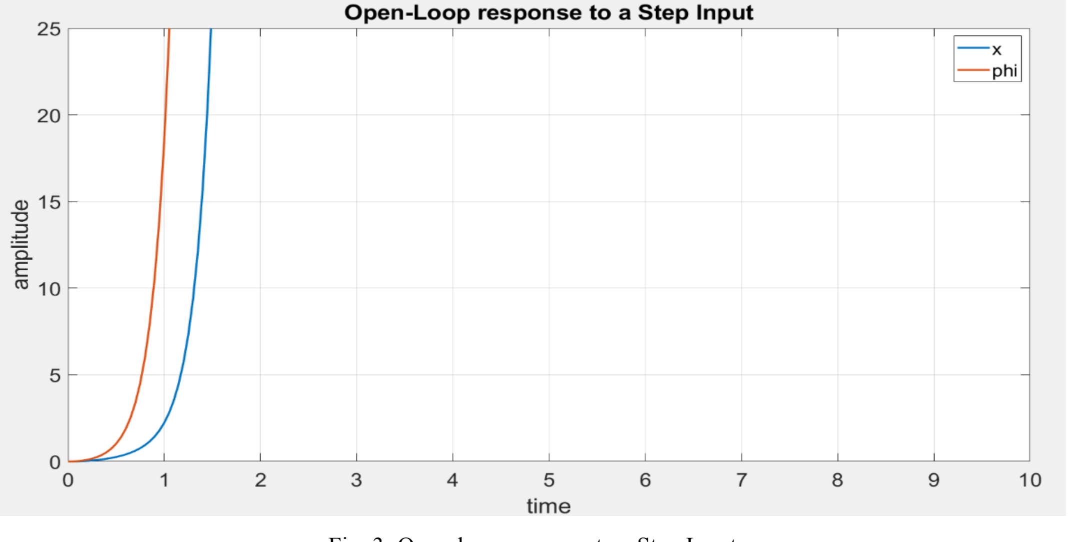

Fig. 3: Open-loop response to a Step Input

3. OPTIMAL CONTROL THROUGH LQR

Optimal control can be defined as the process of determining state trajectories for a system that varies with time to minimize the costs and maximize performance [10]. The main objective here is to strike a balance between the physical constraints and the performance criterion. Hence we require to develop a controller which would cause a dynamic system to reach a fixed target with the given physical limitations.

A Linear Quadratic Regulator (LQR) is such an optimal control method that provides a robust output while considering states of the unstable system, taking into account the states of the dynamical system [10], [11]. From previously defined (17), the linear state space equation [16] can be written as

$$\dot{x} = Ax + Bu \tag{24}$$

where $x = [x \quad \dot{x} \quad \phi \quad \dot{\phi}]^T$

Given that the feedback control $u = -Kx$, (24) can be re-arranged as

$$\dot{x} = (A - BK)x \tag{25}$$

Here K is obtained from the minimization of the cost function (26)

$$J = \int (x^T Q x + u^T R u) \tag{26}$$

$$K = R^{-1} B^T P \tag{27}$$

Q is defined as a semi-definitive symmetric constant matrix and R is a positive definitive symmetric constant matrix in (26) and P in (27) is also a positive definitive symmetric constant matrix derived from algebraic Riccati Equation (26).

$$A^T P + PA - P^2 B^{T+1} R^{-1} + Q = 0 \tag{28}$$

On further examination of our system, we find that it is controllable and observable through equations (29) and (30).

$$C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \tag{29}$$

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{30}$$

3.1 Designing the controller

To find the feedback gain vector K, we use the linear quadratic regulation method. The R and Q parameters which we select will define the balance between the relative importance of the control effort and the deviation from the central position of the pendulum.

For the initial case, we will assume $R = 1$ and $Q = C'C$. Here we give equal importance on both control parameters (cart's velocity and output variables (ϕ)).

$$Q = C'C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

Here Q (1,1) and Q (3,3) are the weights of the cart position and the pendulum angle, respectively.

What matters to us is the relative difference between the values of Q and R and not their absolute values. For our problem, we set Q (1,1) to 5000 and Q(3,3) to 100.

$$Q = \begin{bmatrix} 5000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

While it may appear drastic to compromise the pendulum deviation angle for the cart position control, it is found that the settling time for the pendulum is still within our design parameters.

Such a configuration yields a gain matrix of $K = [-223.6068 \quad -107.1604 \quad 258.0306 \quad 51.4370]$.

It is observed that a steady-state error exists in the cart position which can be eliminated by changing the reference input signal itself.

In the system described, all the state variables are fed back into the controller, thus a need arises to find a constant value that will be added to our feedback after multiplying it with the feedback gain array *K*.

This can be accomplished by adding a pre-compensating or a scaling constant to our reference denoted by K_{ref}.

For our case, K_{ref} is calculated to be -223.6068 using the '*rscale*' algorithm [17]. Fig. 4 shows perfect behavior as expected after adding the scaling parameter to the reference input.

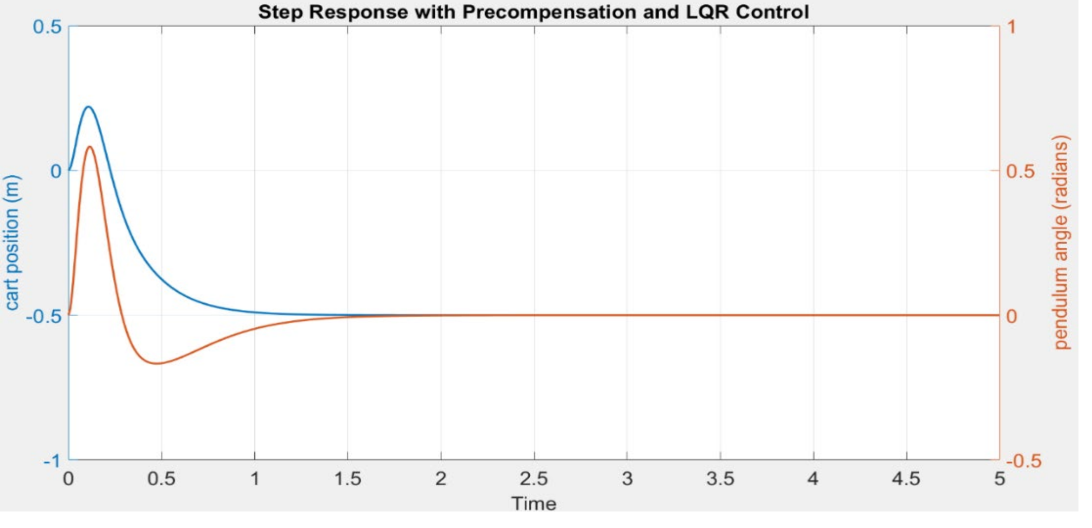

Fig. 4: Step response with K_{ref} applied

3.2 Designing an observer-based control

An observer-based controller comprises a real-time simulation making use of a correction term along with the same input as the plant (33), [12]. L is the observer gain matrix and $\hat{y}$ is an estimate of the plant's output.

The observer gain matrix is used to correct the state estimate based on the difference between the actual and the estimated output.

It is required that the observer poles be faster than the controller poles to make the state estimate converge faster. The controller poles are found to be (34).

$$\delta\hat{x} = A\dot{x} + Bu - L(y - \hat{y}) \tag{33}$$

$$p_C = \begin{bmatrix} -14.8355 + 14.5151i \\ -14.8355 - 14.5151i \\ -4.7391 + 0.8146i \\ -4.7391 - 0.8146i \end{bmatrix} \tag{34}$$

By placing the estimator poles at a higher value than our slowest pole (-4.7391), we can obtain the estimator gain matrix (K_e) in a similar algorithm developed for finding feedback gain (K) stated previously and then applying a step input Fig. 5, a stable system is created using both (Φ) and (x)

$$K_e = \begin{bmatrix} 82.6415 & -1.0371 \\ 1.7002 & -40.0023 \\ -1.4865 & 83.3575 \\ -77.0093 & 1.7504 \end{bmatrix} \tag{35}$$

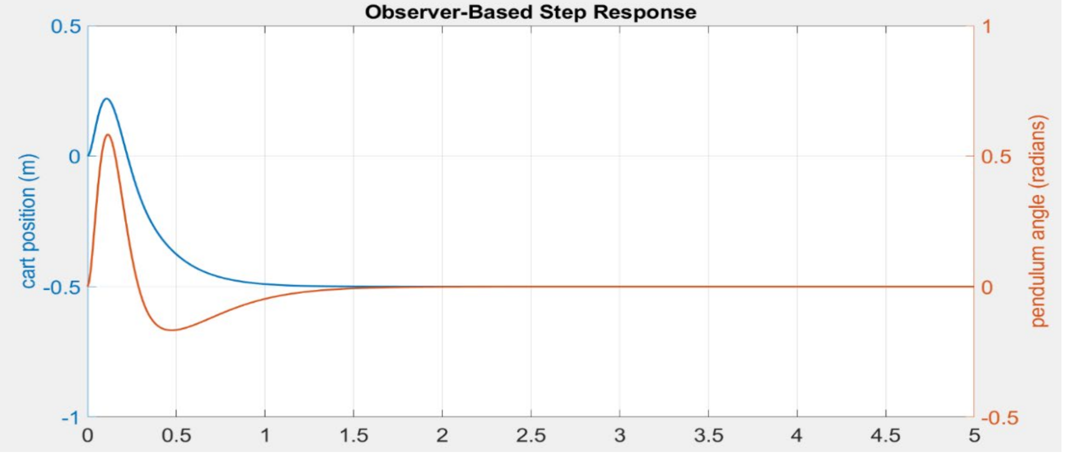

Fig. 5: Step Response with Observer-Based Feedback control

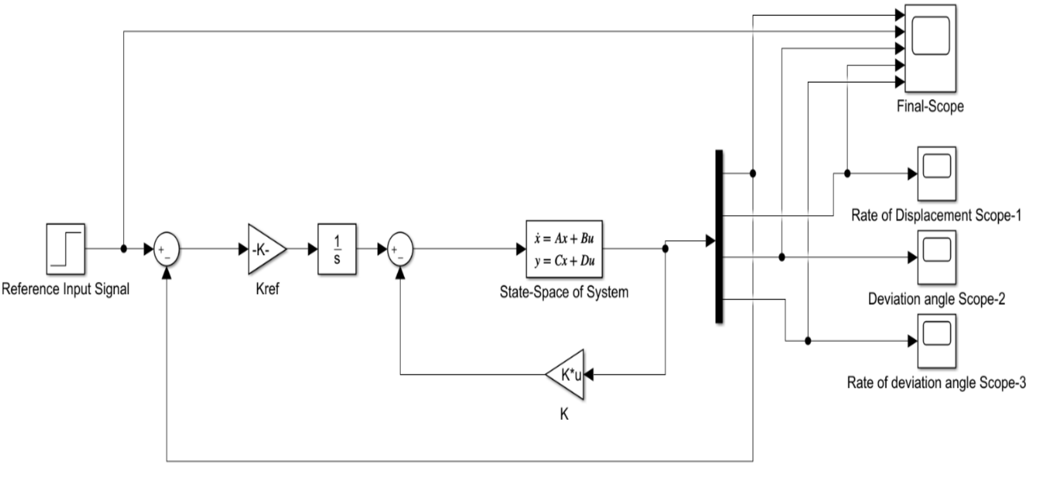

Fig. 6: Designing the system in Simulink, with full state feedback

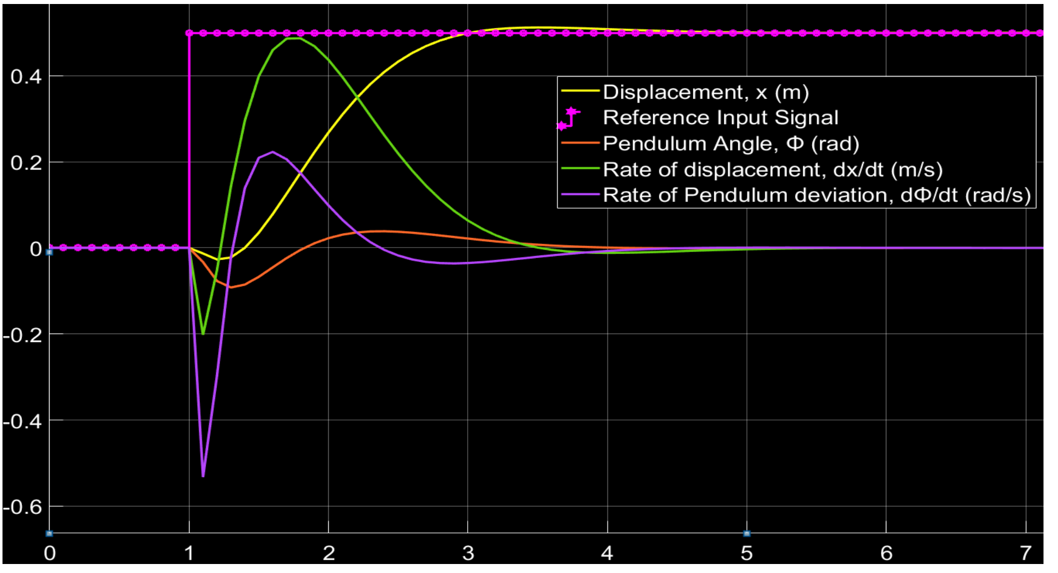

Fig. 7: Unit pulse response of the LQG controller

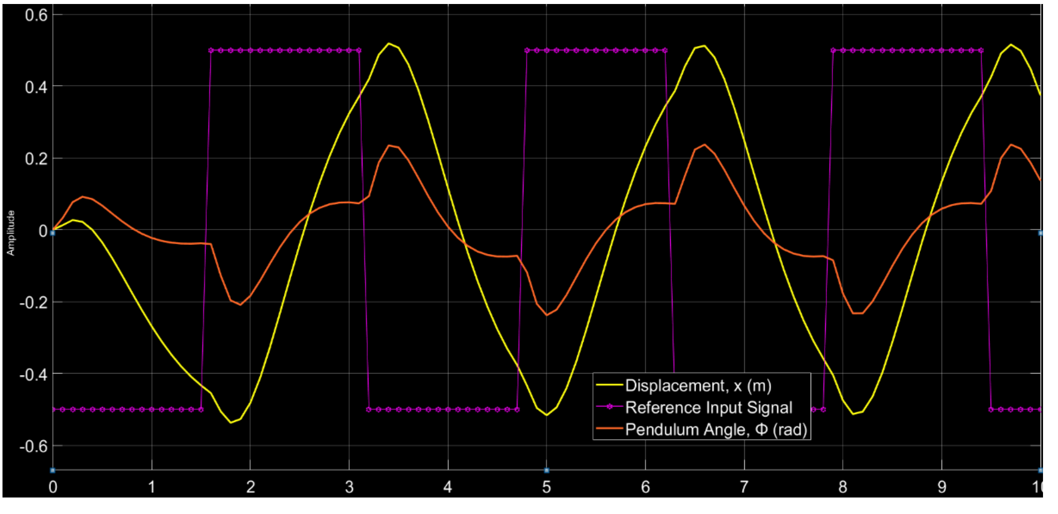

Fig. 8: Continuous pulse response of the LQG controller

4. LINEAR QUADRATIC GAUSSIAN CONTROLLER

A Linear Quadratic Gaussian controller (LQG) is in essence, an LQR controller combined with a Kalman filter. An LQG controller is as dynamic as the system it controls. The schematic of the LQG controller is depicted in Fig. 9. They are applicable to linear time-variant as well as time-invariant systems [13], [14]. The separation principle states that the state estimator and the state feedback are independent of each other. A changed form of (24), (25), and (26) would now be described by:

$$K_e = \begin{bmatrix} 82.6415 & -1.0371 \\ 1.7002 & -40.0023 \\ -1.4865 & 83.3575 \\ -77.0093 & 1.7504 \end{bmatrix} \tag{36}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t) \tag{37}$$

$$y(t) = C(t)x(t) + w(t) \tag{38}$$

$$\dot{\hat{x}} = (A - LC - BK)\hat{x} + Ly \tag{39}$$

$$J = \mathrm{E}[x^T \mathrm{Fx(T)} + \int_0^T x^T(t)Q(t)x(t) + u^T\mathrm{(t)R(t)(t)\ dt}] \tag{40}$$

where $y(t)$ resembles the vector of available outputs for feedback, L is the Kalman gain matrix, $v(t)$ and $w(t)$ are the white Gaussian noise affecting the system and E is the expected value.

We use our previously found value of $K = [-223.6068 \quad -107.1604 \quad 258.0306 \quad 51.4370\,]$ directly in the new controller design.

A Kalman filter is a set of equations that are used for minimizing the estimated error when the variables are linearly transformed provided that some pre-requisite conditions are met [15].

The Kalman gain is given by

$$L(t) = S_e(t)C^T R^{-1} \tag{41}$$

$$S_e = \mathrm{E}[\big(x(t) - \hat{x}(t)\big)^T (x(t) - \hat{x}(t))] \tag{42}$$

To find the Kalman gain matrix, the MATLAB function '*kalman'* was used and was found to be

$$L = \begin{bmatrix} 10.8148 & 0.1679 \\ 8.4945 & 3.6518 \\ 0.1679 & 15.0779 \\ 0.6966 & 63.6857 \end{bmatrix} \tag{43}$$

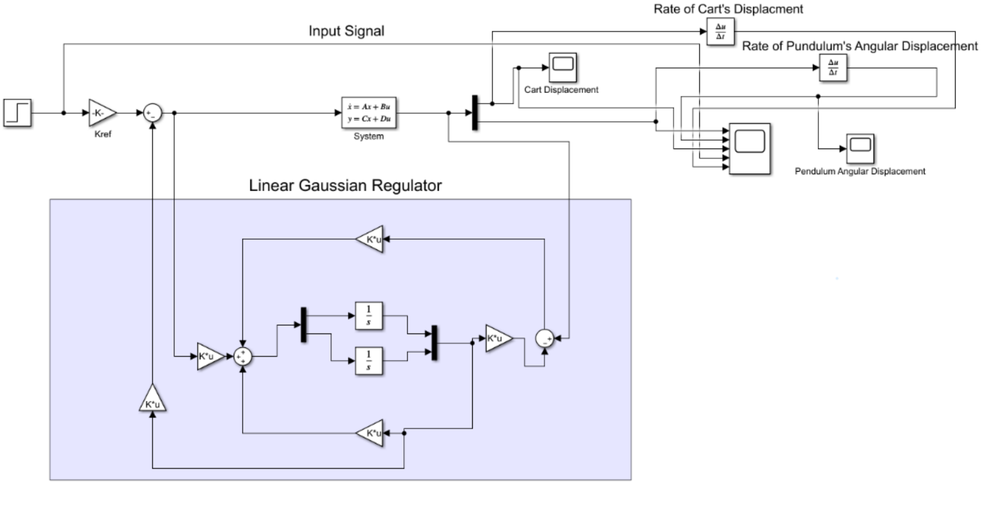

Fig. 9: Simulink Diagram of an LQG controller

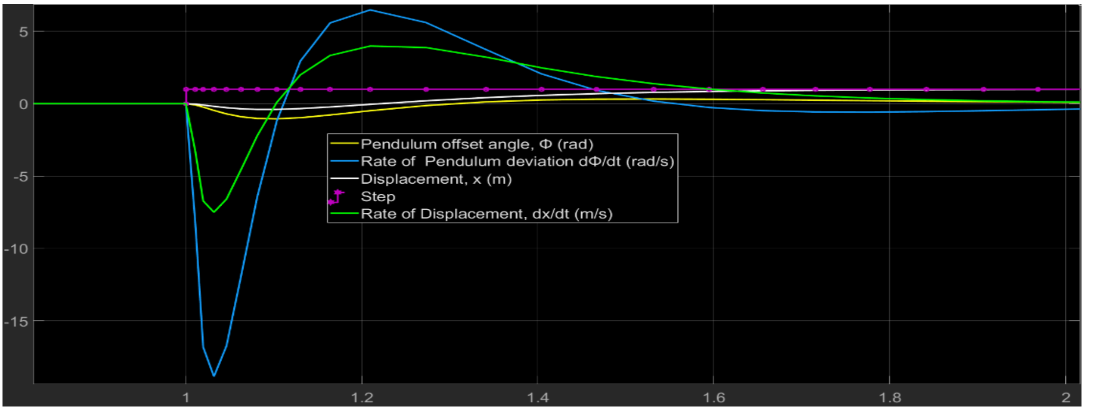

Fig. 10: Simulink output for a Unit Pulse

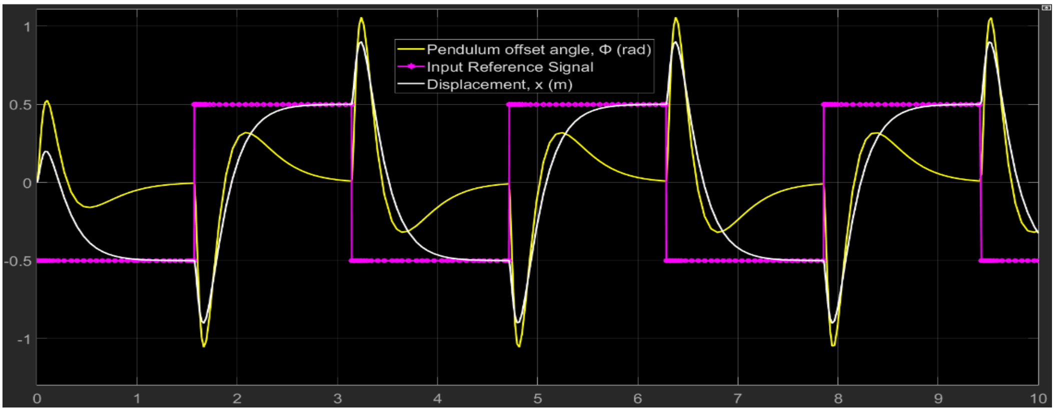

Fig. 11: Simulink output for a Continuous Pulse

5. CONCLUSIONS

From Fig. 7 and Fig. 10 it is clear that there is a significant decrease in settling time (from ≈3 to 1.6), thus proving that an LQR controller is more suitable for such a system in terms of accuracy. However, a huge jump is noticed in the rate of displacement and the rate of angular deviation of the pendulum. There might be circumstances where the hardware does not support such high rates and may cause malfunctioning of the entire system itself. On the other hand, while the LQR controller is slower in response, it significantly relaxes the rate of change.

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