The Relative Inertiality - The Relative Mechanical Movement of Bodies in Outer Space

Sorin Stefan RADNEF*

*Corresponding author
INCAS – National Institute for Aerospace Research "Elie Carafoli",
Bd. Iuliu Maniu 220, Bucharest 061126, Romania,
radnefss@yahoo.com

DOI: 10.13111/2066-8201.2025.17.2.10

Received: 23 February 2025/ Accepted: 26 May 2025/ Published: June 2025

Abstract: *Even though classical mechanics states that it is not possible to find an inertial reference frame, relative measurements are used to determine the mechanical behaviour of celestial bodies with very small dimensions compared to the distances between them. The present work aims to demonstrate that the two fundamental theorems of mechanics, the momentum theorem and the angular momentum theorem, remain valid if we use the relative distances and velocities between these celestial bodies. The starting point for developing such a result is that the basic assumptions of Newtonian mechanics are considered to be valid and are used with the assumption that an inertial reference frame can exist, even if only hypothetically. Relationships are established between the relative accelerations and the global forces acting on each body in a two-body ensemble, which are quite similar to those stated in an inertial reference frame.*

Key Words: *inertial reference frames, relative mechanical motion, general equation of relative motion*

1. INTRODUCTION

This paper is based on a previous communication given at the "Aerospace Conference" held at INCAS – Bucharest in 2022, [1]. According to classical (Newtonian) mechanics, [2], [3], [4], there cannot be an inertial reference system, which we will denote as (SRIn), i.e. a reference system in which the principles of this mechanics are valid. What can be determined are classes of reference systems in which the laws of mechanical motion are identical, using the principles of Newtonian mechanics and assuming that such a system exists. These are the reference systems moving rectilinearly and uniformly relative to each other. Since an absolute (SRIn) is not accessible, the only kinematic determinations that we can practice for using of these principles are those regarding the relative mechanical motion between point-like bodies, i.e. those bodies for which their own geometric dimensions are unimportant in their relative motion, and they can be assimilated to some geometric points.

In outer space, kinematic determinations are typically used to analyse the relative motions between cosmic bodies [5], and are then often used as if they were established in an inertial reference. The results deduced in this way are confirmed with sufficient accuracy by the experimentally determinations. It thus becomes necessary to validate such a physical situation in the same context in which the kinematic measurements take place and the way in which they are used. Therefore, the objective of interest is to verify whether and how the relative motions between point bodies, can be analyzed independent of an (SRIn), in a similar way to the situation in an (SRIn).

We consider a mechanical ensemble of n point bodies, denoted $C_{k=\overline{1,n}}$, which is highlighted by reference to the other bodies with which they interact. This mechanical system is denoted (SMIn). For the geometry of the relative mechanical movement, the "Euclidean geometry" will be considered valid. The analysis of the relative motions in this (SMIn) is done using the kinematics and inertial properties corresponding to a (SRIn) that we assume to exist, so we use the concepts, principles and theorems of Newtonian mechanics.

It is stated, at the outset, that the intrinsic description of relative motions is not the subject of this communication, and no attention is paid to the fundamental principles and ideas concerning the basic physics of mechanical motion. Attention is restricted to the classical principles of mechanics regarding the relative motion of point-like bodies (which we name from now on point bodies) interacting with each other as well as with other external bodies with respect to (SMIn).

The achievement of the proposed general objective is obtained by successively passing through several clear stages:

1. Mechanical analysis of bilateral interactions and their inertial properties, in the case of n=2
2. Determining the structure of the actions exerted on a point body located in a (SMIn) with n≥3
3. Evaluation of the resultant force applied to a point body in (SMIn) using the bilateral interaction forces coming from the relative motion
4. Validation of the two fundamental themes of the dynamics using the bilateral interactions forces resulting from the relative motion
5. General equation for relative motion in a (SMIn)

Classical books on newtonian mechanics, such as references [2], [3], [4], do not pay attention to the connections between mechanical motions in a (SRIn) and relative motions of bodies in a (SMIn), so that as to put emphasize the deep relationships between them. The present work aims to outline a way of analysing motions for which there is no knowledge of the actual interactions and their representative forces, but for which we can infer the properties of their relative motions as revealed by actual measurements of the relative kinematics.

2. THE ELEMENTARY BILATERAL INTERACTION

The main goal is to analyse the intrinsic mechanical behaviour of (SMIn), that is using of physical quantities of a mechanical nature that depend exclusively on the bilateral relative mechanical motion/ state between each 2 bodies in the considered mechanical system. For this purpose it is necessary to highlight the properties of the relative mechanical movement when there are only 2 bodies, denoted as C_1 and C_2, in the system chosen for the mechanical study. The study sheds light on the properties of bilateral interaction, which are the primary characteristics by which interactions are generally analysed.

We formulate the hypothesis that there is an (SRIn). In such an (SRIn), the relative motion of the two bodies is usually analyzed around the center of mass (in fact the inertia center of these bodies), denoted here by C. The geometric scheme of such a two-point system of bodies is as in the following image. The basic quantitative relationships are stated as known:

$$\vec{A}_1 = \ddot{\vec{R}}_1 \;\;\&\;\; m_1\vec{A}_1 = \vec{F}_1 \;\;\&\;\; \vec{A}_2 = \ddot{\vec{R}}_2 \;\;\&\;\; m_2\vec{A}_2 = \vec{F}_2 \tag{1$_1$}$$

$$\vec{A}_C = \ddot{\vec{R}}_C \ \ \& \ \ (m_1 + m_2)\vec{A}_C = \vec{F}_1 + \vec{F}_2$$

$$m \stackrel{\text{def}}{=} m_1 + m_2 \ \ \& \ \ \vec{R}_1 = \vec{R}_C + \vec{r}_{1C} \ \ \& \ \ \vec{R}_2 = \vec{R}_C + \vec{r}_{2C} \ \ \& \ \ \vec{r}_{12} = \vec{R}_1 - \vec{R}_2 = \vec{r}_{1C} - \vec{r}_{2C}$$

$$\vec{r}_{1C} = \frac{m_2}{m} \cdot \vec{r}_{12} \ \ \& \ \ \vec{r}_{2C} = -\frac{m_1}{m} \cdot \vec{r}_{12} \tag{1_2}$$

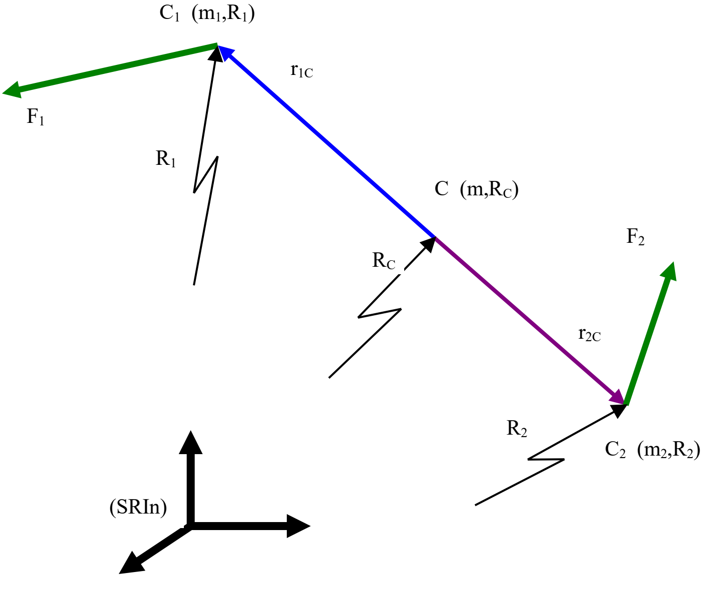

Fig. 1 Two body interaction in (SRIn)

In the following, some generic notations are used:

- the physical quantities in (SRIn) are capitalized
- subscripts are the order numbers of the bodies or the characteristic letters for the corresponding physical quantity
- in lower case are noted the physical quantities in (SMIn), which are physical quantities characteristic of the relative movements between the bodies in (SMIn)
- the vectors will be denoted in text with their letters only
- *R* are the vector radii in (SRIn); *r* are the relative vector radii
- *V* are the speeds in (SRIn); *v* are the relative velocities in (SMIn)
- *A* are the accelerations in (SRIn), *a* are the relative accelerations in (SMIn)
- *m* are the masses in (SRIn); as well as relative ones in (SMIn)

With these clarifications we will have the relations in (SRIn):

$$\vec{a}_{1C} = \vec{A}_1 - \vec{A}_C \ \ \& \ \ \vec{a}_{2C} = \vec{A}_2 - \vec{A}_C$$

$$m_1\vec{a}_{1C} = \vec{F}_1 - \frac{m_1}{m}(\vec{F}_1 + \vec{F}_2) \ \ \& \ \ m_2\vec{a}_{2C} = \vec{F}_2 - \frac{m_2}{m}(\vec{F}_1 + \vec{F}_2) \tag{2}$$

$$\vec{a}_{12} = \vec{A}_1 - \vec{A}_2 = \vec{a}_{1C} - \vec{a}_{2C}$$

$$\frac{m_1 m_2}{m}\vec{a}_{12} = \frac{m_2}{m}\vec{F}_1 - \frac{m_1}{m}\vec{F}_2 \qquad \& \qquad \frac{m_1 m_2}{m}\vec{a}_{21} = \frac{m_1}{m}\vec{F}_2 - \frac{m_2}{m}\vec{F}_1 \tag{3}$$

Stating the next notations:

$$\vec{F}_{12} = \frac{m_2}{m}\vec{F}_1 - \frac{m_1}{m}\vec{F}_2 \quad \& \quad \vec{F}_{21} = \frac{m_1}{m}\vec{F}_2 - \frac{m_2}{m}\vec{F}_1 \quad \text{cu:} \quad \vec{F}_{12} = -\vec{F}_{21} \tag{4}$$

we will derive the following formulas:

$$m_1\vec{a}_{1C} = \vec{F}_{12} \quad \& \quad m_2\vec{a}_{2C} = \vec{F}_{21} \tag{5$_1$}$$

$$\frac{m_1 m_2}{m}\vec{a}_{12} = \vec{F}_{12} \quad \& \quad \frac{m_1 m_2}{m}\vec{a}_{21} = \vec{F}_{21} \tag{5$_2$}$$

The situation is equivalent from the point of view of relative movement to replacing some general forces in (SRIn), F_1 and F_2, applied to bodies C_1 and C_2 with the equal and opposite forces F_{12} and F_{21} that may be determined using kinematical relationships in (SMIn). Thus, the relative motion of the two point bodies in the inertial reference system under the actions represented by the forces F_1 and F_2 is equivalent to the relative motion determined by the relative kinematics in the mechanical system (SMIn) and in which the bodies are under the action of a bilateral interaction represented only by the forces F_{12} and F_{21}. These forces can be determined in (SMIn) from the relative motion, analogously to the forces determined from the absolute motion detected in (SRIn).

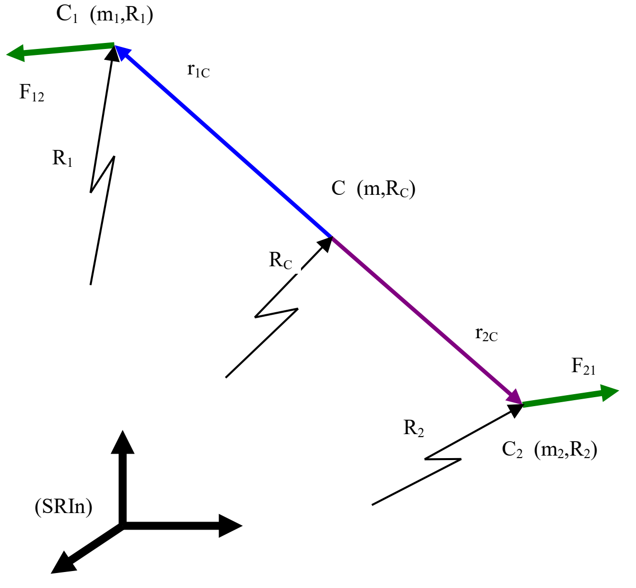

Fig. 2 The two bodies as they interact in (SMIn)

From this equivalence it may be inferred that the mechanics of the relative motion of the two bodies is a problem in its own right because it is expressed only in terms of intrinsic physical quantities. It becomes natural to express the relative momentum of the motion, denoted as $\boldsymbol{H}_{12}$ and its rate of change as measures the relative forces that can be revealed between the two bodies. In this respect and considering the relative kinematics in (SMIn) we must have the main relationship to define the inertia properties revealed in the relative motion, represented by the corresponding relative mass, denoted $\boldsymbol{m_r}$, and understood according to the mechanics principles:

$$m_r(\vec{A}_1 - \vec{A}_2) = \dot{\vec{H}}_{12} \stackrel{\text{def}}{=} F_{12}^{rel} \tag{6$_1$}$$

On the other hand, the dynamic phenomena in (SRIn) and (SMIn) must be revealed in the same way and have the same meaning in both reference systems. So, we have the following relationships:

$$m_r\left[\left(\dot{\vec{V}}_1-\dot{\vec{V}}_C\right)-\left(\dot{\vec{V}}_2-\dot{\vec{V}}_C\right)\right]\equiv\dot{\vec{H}}_{12}$$
$$m_1\left(\vec{V}_1-\vec{V}_C\right)-m_2\left(\vec{V}_2-\vec{V}_C\right)\stackrel{\text{def}}{\equiv}\vec{H}_{12} \tag{6_2}$$

$$m_1\vec{a}_{1C}-m_2\vec{a}_{2C}=\dot{\vec{H}}_{12}\quad\&\quad\dot{\vec{H}}_{12}=\vec{F}_{12}-\vec{F}_{21}\equiv 2\vec{F}_{12} \tag{6_3}$$

where the physical (mechanical) quantities $\vec{r}_{1C},\vec{r}_{2C};\ \vec{v}_{1C},\vec{v}_{2C};\ \vec{a}_{1C},\vec{a}_{2C}$ are *the ones that can be determined wthout referring to an inertial reference frame* (SRIn).

Doing the appropriate substitutions for the relative motion in (SMIn) considering (5_1), we derive:

$$m_r\left(\frac{1}{m_1}\vec{F}_{12}-\frac{1}{m_2}\vec{F}_{21}\right)=\vec{F}_{12}^{rel}\equiv 2\vec{F}_{12}$$
$$m_r\left(\frac{m_1+m_2}{m_1m_2}\vec{F}_{12}\right)=\vec{F}_{12}^{rel}\equiv 2\vec{F}_{12} \tag{7}$$

and so, the relative mass expression from the above equation is:

$$m_r=\frac{2m_1m_2}{m} \tag{8_1}$$

This formula certifies that the relative mass is the harmonic average of the masses of the two body:

$$\frac{1}{m_r}=\frac{\frac{1}{m_1}+\frac{1}{m_2}}{2} \tag{8_2}$$

Now, considering only the relative motion in (SMIn), it is convenient and meaningful to rewrite the relations (5_1), (5_2) as:

$$m_r\vec{a}_{1C}=\vec{F}_{12}^{rel}\quad\&\quad m_r\vec{a}_{2C}=\vec{F}_{21}^{rel} \tag{5_1}$$

$$m_r\vec{a}_{12}=\vec{F}_{12}^{rel}\quad\&\quad m_r\vec{a}_{21}=\vec{F}_{21}^{rel} \tag{5_2}$$

3. THE STRUCTURE OF THE INTERACTION EXERCISED ON A BODY THAT BELONGS TO (SMIn)

Classical mechanics equates the general motion of a system of point bodies (SMIn) with a superposition of two motions:

a. the general motion of all bodies, solidified as a single solid body in translational motion under the resultant action of all interactions exerted on each body in (SMIn), having as generic point the center of mass (which we may call "center of inertia"), denoted as C; thus, this motion is equivalent to the motion of the center of mass that is under the resultant action

b. motion around the inertial center of the ensemble of point bodies, as a one whole system

From the kinematics of the known classical mechanics, the relative motions between bodies from (SMIn) may be represented also as being that relative to the common center of mass. The relative motions are analysed coming from the (SRIn) by using the two foundamntal theorems of dynamics, the momentum theorem and the moment of momentum theorem that are formulated relative to the mass center.
We state from the beginning the following notations, with their significance, that will be used in the followings:

$\vec{F}_k^e$ the resultant force that represents the action exerted on the body C_k from the outside of (SMIn)

$\vec{F}_k^i$ the resultant force acting on the body C_k from the inside of (SMIn)

$\vec{F}_{kl}^i$ the force representing the internal interaction between the bodies C_k and C_l considering their motion in (SRIn)

$\vec{F}_{kl}$ the force representing the bilateral interaction between the bodies C_k and C_l, considering their relative motion in (SMIn), coming from the forces $\vec{F}_k$, $\vec{F}_l$

$\vec{F}_k$ the resultant force applied to C_k body, for its relative motion to the other bodies, in (SRIn)

$\vec{F}_k^0$ the resultant force applied to C_k body, for its relative motion to the other bodies, in (SMIn)

$\bar{C}_k$ the inertia center of all bodies of (SMIn) without C_k body; C the mass center of all bodies in (SMIn)

$\bar{m}_k$ the mass of all bodies of (SMIn) without the C_k body

In (SRIn), the actions of the other point bodies of the system are exerted on the body C_k, having the resultant force:

$$\vec{F}_k = \vec{F}_k^e + \vec{F}_k^i$$
$$\vec{F}_k^i = \sum_{l \neq k}^{n} \vec{F}_{kl}^i \tag{9}$$

and the equation of the dynamic motion:

$$m_k \vec{A}_k = \vec{F}_k \equiv \vec{F}_k^e + \sum_{l \neq k}^{n} \vec{F}_{kl}^i \tag{9$_1$}$$

According to the momentum theorem, all other bodies in (SMIn), as a rigid whole, have a motion composed of:

-the motion as a point body coincident with the center of mass having the acceleration $\vec{A}_{\bar{k}}$:

$$\bar{m}_k \vec{A}_{\bar{k}} = \sum_{l \neq k}^{n} \vec{F}_l \equiv \sum_{l \neq k}^{n} \vec{F}_l^e + \sum_{s \neq l}^{n} \sum_{l \neq k}^{n} \vec{F}_{ls}^i \tag{10}$$

where we have the obvious relation for all internal interactions:

$$\sum_{s\neq l}^{n}\sum_{l\neq s}^{n} F_{ls}^{i} \equiv \vec{0} \tag{10$_1$}$$

and

-its proper motion around the center of mass.

From relations (10) it follows:

$$\bar{m}_k \vec{A}_{\bar{k}} = \sum_{l\neq k}^{n} \vec{F}_l^{e} + \sum_{l\neq k}^{n} \vec{F}_{lk}^{i} \tag{11}$$

Therefore, according to equations (9$_1$) and (11) the interaction between the body C_k with the rest of the bodies in (SMIn) is equivalent to the interaction between the body C_k and the center of mass of the other bodies in (SMIn). We are therefore in the situation of interaction of two point bodies, for which the method of approaching the relative motion was developed.

The relationship that describes the dynamics of the relative motion of the body C_k is:

$$m_r \vec{a}_{k\bar{k}} = \vec{F}_{k\bar{k}}^{rel} \tag{12}$$

in which:

$$\begin{gathered} m_r = \frac{2m_k \bar{m}_k}{m_k + \bar{m}_k} = \frac{2m_k \bar{m}_k}{m} \\ a_{k\bar{k}} = \left(\vec{A}_k - \vec{A}_{\bar{k}}\right) \\ \vec{F}_{k\bar{k}} \equiv \frac{1}{2}\vec{F}_{k\bar{k}}^{rel} = \frac{\bar{m}_k}{m}\vec{F}_k - \frac{m_k}{m}\sum_{l\neq k}^{n} \vec{F}_l \end{gathered} \tag{12$_1$}$$

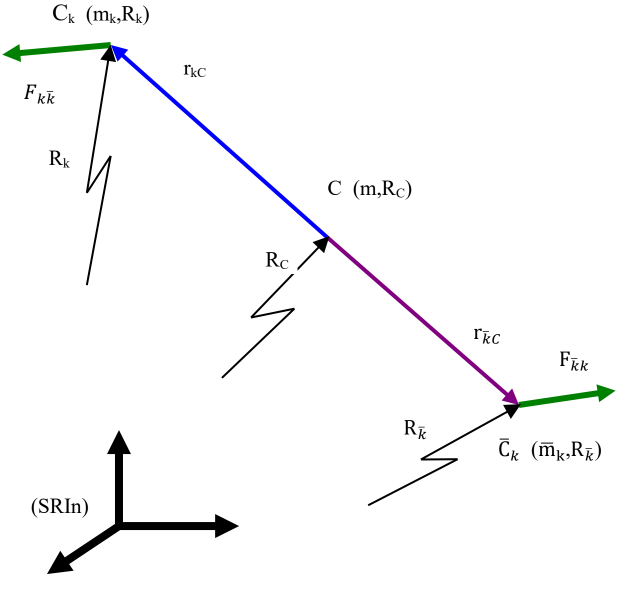

Fig. 3 The C_k body interacting with the all other bodies of (SMIn)

Developing the bilateral force expression $\vec{F}_{k\bar{k}}$ we may write the formula

$$\vec{F}_{k\bar{k}} = \frac{\bar{m}_k}{m}\vec{F}_k - \frac{m_k}{m}\sum_{l\neq k}^{n}\vec{F}_l = \sum_{l\neq k}^{n}\left(\frac{m_l}{m}\vec{F}_k - \frac{m_k}{m}\vec{F}_l\right) \equiv \sum_{l\neq k}^{n}\vec{F}_{kl} \tag{12$_2$}$$

So, the dynamics of the relative motion in (SMIn) of the body C_k, with respect to the rest of bodies from this mechanical system, considered as a rigid entity, has the equation:

$$m_r\vec{a}_{k\bar{k}} = 2\sum_{l\neq k}^{n}\vec{F}_{kl}$$

or: (13)

$$m_k\vec{a}_{kC} = \sum_{l\neq k}^{n}\vec{F}_{kl}$$

which means that the body C_k , in the relative motion in (SMIn), is under the bilateral actions with the other bodies of (SMIn), represented by forces F_{kl}, that may be determined using the relative motions of couples (C_k, C_l) in (SMIn).

4. THE EQUIVALENCE BETWEEN THE RESULTING FORCE AND THE FORCES OF BILATERAL INTERACTION

Considering (SRIn), the component of the relative motion about the common center of mass of all bodies has the following equation of dynamics for the relative motion of C_k with respect to the center of mass C:

$$m_k\vec{a}_{kC} = \vec{F}_k - \frac{m_k}{m}\sum_{l=1}^{n}\vec{F}_l = \sum_{l=1}^{n}\left(\frac{m_l}{m}\vec{F}_k - \frac{m_k}{m}\vec{F}_l\right)$$
$$\vec{a}_{kC} = (\vec{A}_k - \vec{A}_C) \tag{14}$$

The term for $l=k$ vanishes and remains only the summ from the formula (12$_2$). Thus, we have the dynamics equations for the relative motion detected in (SRIn) and in (SMIn):

$$m_k\vec{a}_{kC} = \sum_{l\neq k}^{n}\left(\frac{m_l}{m}\vec{F}_k - \frac{m_k}{m}\vec{F}_l\right) = \sum_{l\neq k}^{n}\vec{F}_{kl}$$
$$m_r\vec{a}_{k\bar{k}} = 2\sum_{l\neq k}^{n}\vec{F}_{kl} \equiv 2\sum_{l\neq k}^{n}\left(\frac{m_l}{m}\vec{F}_k - \frac{m_k}{m}\vec{F}_l\right) \tag{15}$$

Between the relative pozition vectors of the two mass center there is the relationship of definition:

$$m_k\vec{r}_{kC} + \bar{m}_k\vec{r}_{\bar{k}C} = \vec{0} \tag{16$_1$}$$

which leads to the formulas:

$$\bar{r}_{kC} = \frac{\bar{m}_k}{m}\vec{r}_{k\bar{C}_k} \quad \& \quad \vec{r}_{\bar{k}C} = -\frac{m_k}{m}\vec{r}_{k\bar{C}_k} \tag{16$_2$}$$

As a direct consequance of the above formulas the equations (15) are identical for the relative motion in (SRIn) and (SMIn), which means that the forces of bilateral interactions, F_{kl}, detected by relative kinematics in (SMIn), produce the same resultant force as the forces that are detected in (SRIn).

5. THE FOUNDAMENTAL THEOREMS OF DYNAMICS USING THE FORCES OF BILATERAL INTERACTION

It has been shown that the equations of relative motion using the bilateral interaction forces in (SMIn) are, see (14):

$$\frac{d}{dt}(m_k\vec{v}_{kC}) \equiv m_k\vec{a}_{kC} = \sum_{l=1}^{n}\vec{F}_{kl} \equiv \vec{F}_k^0 \tag{17}$$

where $m_k\vec{v}_{kC} \overset{\text{def}}{=} \overrightarrow{H}_k$ is the momentum of C_k point body and considering F_{kk}=0. So one may use the determinations made in (SMIn) without to alter the results furnished by the Momentum Theorem expessed in (SRIn), denoted here as (MT).

According to the definition relationships we have the obvious relation:

$$\vec{0} = \sum_{k}^{n}\sum_{l}^{n}\vec{F}_{kl}$$

and so:

$$\vec{0} = \sum_{k=1}^{n}\vec{F}_k^0$$

$$\sum_{k=1}^{n}\frac{d}{dt}(m_k\vec{v}_{kC}) \equiv \frac{d}{dt}\sum_{k=1}^{n}(m_k\vec{v}_{kC}) = \sum_{k=1}^{n}\vec{F}_k^0 \equiv \vec{0} \tag{18}$$

where $\sum_k m_k\vec{v}_{kC} \overset{\text{def}}{=} \overrightarrow{H}$ is the momentum of all C_k point bodies and which means that, in relation to the relative movements that can be highlighted by the kinematics and relative forces in (SMIn), this mechanical system, as a rigid assembly, has a rectilinear and uniform movement.

The kinetic momentum theorem for the system of bodies is analyzed, for now, only for the case of point bodies that make up (SMIn). Its expression with respect to the common center of mass is in the inertial reference frame (SRIn):

$$\vec{M}_C \overset{\text{def}}{=} \sum_k \vec{r}_k \times \vec{F}_k = \sum_k \vec{r}_k \times m_k\vec{a}_k \equiv \frac{d}{dt}\sum_k \vec{r}_k \times m_k\vec{v}_k \tag{19}$$

where $\sum_k \vec{r}_k \times m_k\vec{v}_k \overset{\text{def}}{=} \vec{K}_C$ is the kinetic momentum of all the point bodies.

We must now prove that, using the kinematic determinations in (SMIn) for the relative motions that also provide the bilateral forces F_{kl}, we derive the same moment of momentum for the whole (SMIn). Doing this, in (SMIn) we have the relationships:

$$m_k \vec{a}_k = \vec{F}_k^0 = \sum_l \vec{F}_{kl} = \sum_l \left(\mu_l \vec{F}_k - \mu_k \vec{F}_l\right) \tag{20}$$

where $\mu_k = \frac{m_k}{m}$, and so we have to determine the sum:

$$\sum_k \vec{r}_k \times \left[\sum_l \left(\mu_l \vec{F}_k - \mu_k \vec{F}_l\right)\right]$$

By conveniently grouping the terms of the double sum, we can obtain:

$$\begin{aligned}\sum_k \vec{r}_k \times \left[\sum_l \left(\mu_l \vec{F}_k - \mu_k \vec{F}_l\right)\right] &= \sum_k \sum_l \left[\vec{r}_k \times \left(\mu_l \vec{F}_k - \mu_k \vec{F}_l\right)\right] = \\ &= \sum_k \sum_l \left[(\mu_k \vec{r}_k) \times \vec{F}_l\right] - \sum_k \sum_l \left[\mu_l \left(\vec{r}_k \times \vec{F}_k\right)\right] = \\ &= \sum_l \mu_l \sum_k \left(\vec{r}_k \times \vec{F}_k\right) - \sum_k (\mu_k \vec{r}_k) \times \sum_l \vec{F}_l\end{aligned} \tag{21}$$

From the definition of the physiscal quantities μ_l și $\mu_k \vec{r}_k$ one may derive directly:

$$\sum_l \mu_l = 1$$

$$\sum_k \mu_k \vec{r}_k = \vec{0}$$

and the result is as follows:

$$\sum_k \vec{r}_k \times m_k \vec{a}_k \equiv \sum_k \vec{r}_k \times \left[\sum_l (\mu_l \vec{F}_k - \mu_k \vec{F}_l)\right] = \sum_k \vec{r}_k \times \vec{F}_k$$

which means that one may use the determinations made in (SMIn) without altering the results provided by the Kinetic Momentum Theorem expessed in (SRIn), denoted here as (KMT):

$$\vec{M}_C \equiv \sum_k \vec{r}_k \times \vec{F}_k^0 = \frac{d}{dt} \vec{K}_C \tag{22}$$

This means that even (KMT) can be expressed using physical quantities determined in relation to the relative motion in (SMIn).

6. THE GENERAL EQUATION FOR THE RELATIVE MOVEMENT

Taking into account relations (5) and (17), the dynamic equation of the relative motion between two bodies in (SMIn) is resumed, highlighting the bilateral interaction forces F_{kl}. Without losing generality, successive relations using exclusively bilateral interactions are deduced as follows for the couple of bodies C_1 and C_2:

$$\vec{a}_{12} = \vec{a}_{1C} - \vec{a}_{2C} = \frac{1}{m_1} \vec{F}_1^0 - \frac{1}{m_2} \vec{F}_2^0 \tag{23}$$

$$m_1 m_2 \vec{a}_{12} = m_2 \vec{F}_1^0 - m_1 \vec{F}_2^0 = m_2 \sum_{k\neq 1}^{n} \vec{F}_{1k} - m_1 \sum_{l\neq 2}^{n} \vec{F}_{2l}$$

$$m_1 m_2 \vec{a}_{12} = \sum_{k\neq 1}^{n} m_2 \vec{F}_{1k} - \sum_{l\neq 2}^{n} m_1 \vec{F}_{2l} =$$

$$= m_2 \vec{F}_{12} - m_1 \vec{F}_{21} + \sum_{\substack{k\neq 1\\ k\neq 2}}^{n} m_2 \vec{F}_{1k} - \sum_{\substack{l\neq 2\\ l\neq 1}}^{n} m_1 \vec{F}_{2l}$$

$$m_1 m_2 \vec{a}_{12} = (m_1 + m_2 + m_3 + \cdots + m_n)\vec{F}_{12} - (m_3 + \cdots + m_n)\vec{F}_{12} +$$

$$+ \sum_{\substack{k\neq 1\\ k\neq 2}}^{n} m_2 \vec{F}_{1k} - \sum_{\substack{l\neq 2\\ l\neq 1}}^{n} m_1 \vec{F}_{2l} = m\vec{F}_{12} + \sum_{\substack{k\neq 1\\ k\neq 2}}^{n} m_2 \vec{F}_{1k} + \sum_{\substack{l\neq 2\\ l\neq 1}}^{n} m_1 \vec{F}_{l2} + \sum_{k=3}^{n} m_k \vec{F}_{21}$$

For the last two terms one do the adequate gruping of three bilateral forces and we obtain the formula:

$$m_1 m_2 \vec{a}_{12} = m\vec{F}_{12} + \sum_{\substack{k\neq 1\\ k\neq 2}}^{n} (m_2 \vec{F}_{1k} + m_1 \vec{F}_{k2} + m_k \vec{F}_{21}) \tag{24}$$

The last term represents the contribution of each body in (SMIn) to the relative motion between C_1 and C_2, with the first term representing their main bilateral interaction. We will denote:

$$\vec{F}_{k21} = (\frac{m_2}{m}\vec{F}_{1k} + \frac{m_1}{m}\vec{F}_{k2} + \frac{m_k}{m}\vec{F}_{21}) \equiv \frac{1}{2}\vec{F}_{k21}^{rel} \tag{25}$$

and the relatioship (24) is rewrite as:

$$\frac{m_1 m_2}{m}\vec{a}_{12} = \vec{F}_{12} + \sum_{\substack{k\neq 1\\ k\neq 2}}^{n} \vec{F}_{k21}$$

or: (26)

$$m_r \vec{a}_{12} = \vec{F}_{12}^{rel} + \sum_{\substack{k\neq 1\\ k\neq 2}}^{n} \vec{F}_{k21}^{rel}$$

This is the general equation of relative dynamics in (SMIn). For the “TBP” and for the “Problem of the n Bodies”, denoted as “NBP” the equation (26) give us a structure that implies only bilateral interactions of gravitational type.

The configuration of forces in the complementary term is of the following type:

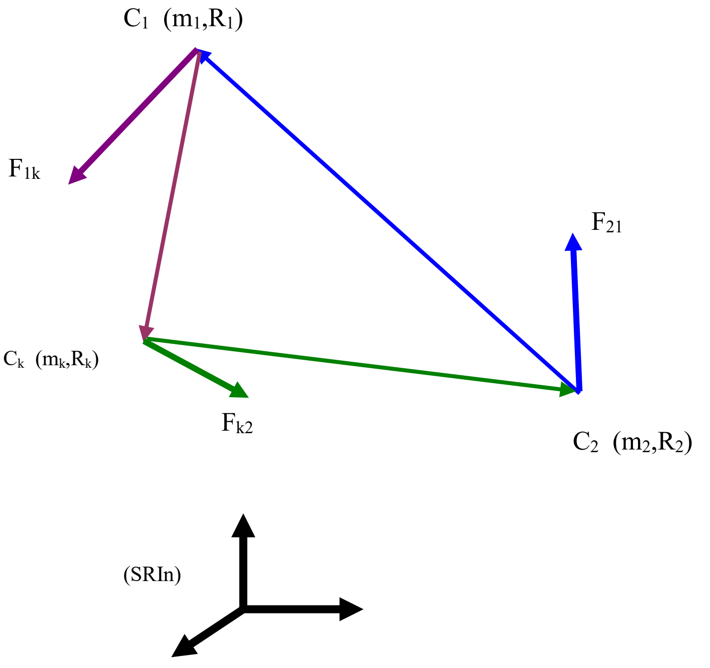

Fig. 4 Bilateral interaction with the influence of the third body C_k

having in a general manner a moment of momentum not zero for the relative motion. From this view point this term will implicitely provide a velocity spectrum with a high probability of being of a turbionaire type, at least localy.

7. CONCLUSIONS

Following the above, it is possible to formulate some conclusions regarding relative motion, as it is intrinsically revealed in a (SMIn).

1. Choosing a mechanical system made up of *n* bodies, we can consider that its internal determinations, i.e. the mechanical determinations corresponding to relative motions, are objective and consistent with the fundamental theorems of mechanics; the momentum theorem and the kinetic momentum theorem will be formally expressed as in an inertial reference frame.

2. The study highlights the way in which bilateral interaction is involved in the relative mechanical motion intrinsically revealed in (SMIn), under the assumptions that the means of kinematics determination are usable and identical as in (SRIn)

3. The motion relative to the center of mass, *C*, can be determined using the bilateral interaction forces determined from the kinematics of relative motions without explicitly and directly knowing the external actions exerted on each body and the effective interactions between them as would be revealed in (SRIn). Therefore, (MT) and (KMT) have the expression:

$$\frac{d}{dt}(m_k\vec{v}_k) = \overrightarrow{F_k^0}$$

$$\frac{d}{dt}\sum_k (\vec{r}_k \times m_k \vec{v}_k) \equiv \sum_k \vec{r}_k \times \vec{F}_k^0$$

4. If we consider the relation (5_1), (5_2), (12) of the relative motion of a body in (SMIn) we will be able to determine the common joint motion of all bodies, at least relative to another body (or even a system of bodies) by including this new body in a new mechanical system of point bodies.

5. The formulas for establishing the dependence of the bilateral interaction forces, F_{kl}, on the resultant forces F_k and F_l from (SRIn) allow determining the relationship of this interaction with that established only for the two bodies (in which the corresponding resultants from (SRIn) are used). From the original definition relationship:

$$\vec{F}_{kl} = \frac{m_l}{m_k + m_l}\vec{F}_k - \frac{m_k}{m_k + m_l}\vec{F}_l$$

the bilateral interaction force necessary to represent the relative motion in (SMIn) is obtained using the bilateral interactions between all bodies of the mechanical system considered

$$\vec{F}_{kl} := \frac{m_k + m_l}{m}\vec{F}_{kl}$$

In the case of a (SMIn) with many point bodies, the forces F_{kl} tend to become increasingly smaller in magnitude, but they are increasingly numerous.
The tendency of the resultant of all these bilateral interaction forces within the entire (SMIn) is obtained using the initial formula that generated the structure of the bilateral interaction forces. Thus, in the equation:

$$\frac{m_k \bar{m}_k}{m_k + \bar{m}_k}\vec{a}_{k\bar{k}} = \sum_{l\neq k}^{n} \vec{F}_{kl} = F_k - \frac{m_k}{\bar{m}_k}\sum_{l\neq k}\vec{F}_l$$

the limit is reached for $\bar{m}_k \overset{(n)}{\rightarrow} \infty$. In this situation, the center of mass of all bodies except C_k tends to merge with the center of mass of the entire mechanical system. Also taking into account the obvious relationships:

$$\lim_{\bar{m}_k \to \infty} \frac{m_k \bar{m}_k}{m_k + \bar{m}_k} = m_k$$

$$\lim_{\bar{m}_k \overset{(n)}{\to} \infty} \sum_{l\neq k}^{n} \vec{F}_{kl} = \lim_{\bar{m}_k \overset{(n)}{\to} \infty} \left(F_k - \frac{m_k}{\bar{m}_k}\sum_{l\neq k}\vec{F}_l \right) = \vec{F}_k$$

under the assumption that the sum $\sum_{l\neq k}\vec{F}_l$ is finite, we arrive at the formula for the dynamics of motion:

$$m_k \cdot \vec{a}_{kC} = F_k$$

which constitutes the expression of the “principle of force” in Newtonian mechanics, expressed with respect to the center of mass of the Univers, here called as (SMIn); in which Univers there are no external actions exerted on the bodies that compose it and therefore has a solidary and uniform movement of the bodies as a whole, thus becoming a referential of type (SRIn) for the relative movements observed between the bodies that compose it.

6. All the above considerations, regarding the results of the present study, allow us to conclude that the relative motion determined exclusively in a system (SMIn) has inertial properties as in an inertial reference system. We can call this “Relative Inertiality”.

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