Cartesian coordinates of an intrinsically defined curve

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Abstract: *Summarizing some old research on the dynamics of a pointall body along its own trajectory, this paper established the differential relationships between the principal curvatures of a 3D curve, that is the normal curvature and the torsional curvature, and its Cartesian coordinates. The differential system thus derived is actually a dynamical system of a representative point of the curve moving along it. This dynamic system is analyzed to see the possibilities of finding analytical solutions in finite terms, using Frobenius' integrability theorem for the general case and usual integration methods for the particular case consisting of the constant ratio between the two curvatures.*

Key Words: *intrinsically defined curves, dynamic system for cartesian coordinates, analitical solutions*

1. INTRODUCTION

A 3D curve is defined intrinsically if it is known its principal curvatures, that is normal curvature denoted by k_n and torsion curvature denoted by k_t, as functions of the arc length, denoted by s, measured prom a reference point of this curve, [1]. This kind of definition is coordinate system independent and so it is more appropriate to the mechanical principles which are dependent only on the physical referential. Because all the mechanical properties of a moving pointall body are usually stated with respect to a coordinate system attached to the physical referential [2], [3], it becomes interesting and necessary to find the cartesian coordinates of such a curve intrinsically defined. The main idea of this paper is to consider the Serret-Frenet reference trihedron attached to a moving point on the curve and so to have a relationship between the angular coordinates of this frame and the principal curvatures [4], considering also the Serret-Frenet differential relationships concerning these curvatures. Till now, only for two dimensional curves there is known the Euler solution for the natural equations considering the angular coordinates of a curve intrinsically defined, [8].

For a general movement of a rigid body, or for a special movement of an aircraft, a differential model which refer to the movement itself of the body and to the intrinsic features of the movement may be more significant for some movement properties, [5-7]. The mechanical movement is governed by physical laws that refer only to the physical referential and not to a coordinate system and so it is desirable to have even the control laws of the required movement not dependent on a coordinate system, but only on the body movement itself. By consequence, the intrinsic definition of the trajectory and the Serret-Frenet frame used as o moving reference trihedron are appropriate to be used considering the previous remarks. On the other hand, the main properties of the plan of flight may be derived by determining the shape and position of the trajectory using the principal curvatures, which are independent of a coordinate system defined on the mechanical movement referential. The advantage of utilizing such intrinsic coordinates is a complete autonomous navigation, without the need of a coordinate system. In parallel, however, the validation of the evolution from a mechanical point of view is done in relation to the movement referential and therefore comparisons are necessary between the coordinates established on board the vehicle (using the principal curvatures of its trajectory) and those established directly on the reference of the movement. This therefore means that it is necessary to be able to determine the (Cartesian) coordinates of the curve that represent the trajectory in a coordinate system attached to the physical reference frame.

2. DIFFERENTIAL EQUATIONS FOR NATURAL PARAMETRIC DEFINED CARTESIAN COORDINATES

The generation of a curve may be conceived as being the trajectory travelled by a pointall body. The properties that identify the curve in any small neighborhood of its every point, without any reference to an external coordinate system, constitute the intrinsic definition of a curve. These properties are the local curvatures of the curve, that is:

- *normal curvature,* k_n
- *torsion curvature,* k_t

If there are known the functions:

$$K_\mathrm{n}: \mathbf{R}_+ \to \mathbf{R}_+\,;\ \forall\, s \in \mathbf{R}_+ \ \to\ k_\mathrm{n} = K_\mathrm{n}(s)$$
$$K_\mathrm{t}: \mathbf{R} \to \mathbf{R}\,;\ \forall\, s \in \mathbf{R} \ \to\ k_\mathrm{t} = K_\mathrm{t}(s) \tag{1}$$

where the independent variable, s, is the curve arc length, measured from a given point as reference, then the curve is intrinsically defined by its own curvatures. Considering the Serret-Frenet formulas:

$$\frac{d\vec{t}}{ds} = k_n\vec{n} \qquad \frac{d\vec{n}}{ds} = k_t\vec{b} - k_n\vec{t} \qquad \frac{d\vec{b}}{ds} = -k_t\vec{n} \tag{2}$$

with $\vec{t}$, $\vec{n}$, $\vec{b}$ the unit vectors along the axes of Serret-Frenet frame, there is an existence proof, that the cartesian coordinates may be inferred [1], using an explicit corresponding ordinary differential system for the director cosines and then integrating the diferential equation of the tangent unit vector to obtain these coordinates. This way, we need another algebraic step to obtain the angular position of the Serret-Frenet trihedron which is necessary as a movement reference frame for mechanics momentum theorem [3], involving difficulties including the sign of these angular variables. Another difficulty regards the linear differential system having variable coefficients which must be analyzed to see if there are or not analytical solutions in finite terms for it. At a glance, there are not such solutions for a curve in 3D, but it must be proven. The only known solution is derived for curves situated in a plane by L. Euler using the natural equations, [11]. So, this way is almost cumbersome and it is necessary to think for another way of bringing the angular variables of the Serret-Frenet frame into the main differential system that state the cartesian coordinates.

An insight into the relationships between the intrinsic properties of a curve and some other variables that describe the relations of the common cartesian coordinates and the curvatures k_n, k_t comes from the kinematics of the motion of the Serret-Frenet frame, denoted by (TdF) with axes (Cx_f), (Cy_f), (Cz_f), along the curve.

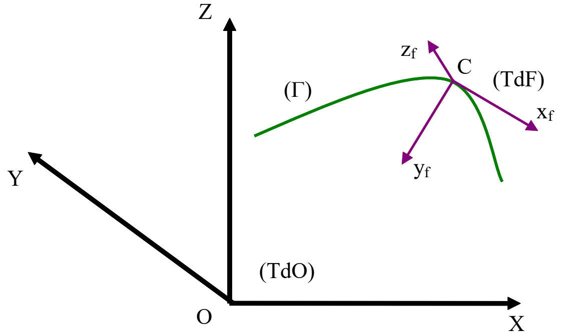

Fig. 1 – The Serret-Frenet frame and the Cartesian Coordonate frame

The variables that are significant for the movement of this reference frame are the attitude angles regarding a cartesian coordinate frame (OXYZ). In the diagram below the origin C coincide with the origin of the cartesian coordinates O, only to emphasize those angles:

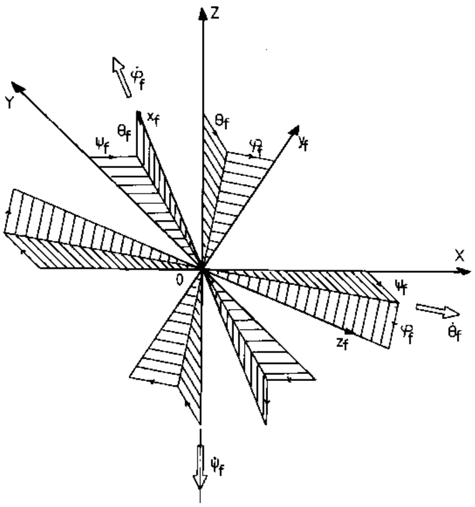

Fig. 2 – The angular coordinates of TdF with respect to (OXYZ)

The meaning of this angles are successive rotations around the axes (Cy_f), (Cz_f) and respective (Cx_f), with usual values as $\psi\in[-\pi,+\pi]$, $\theta\in[-\pi,+\pi]$, $\varphi\in[-\pi,+\pi]$. The unit vector of these rotation directions are as follows, when expressed in (TdF):

$$\begin{aligned}
\vec{u}_\psi &= (-\sin\theta_f)\vec{t} + (-\cos\theta\cos\varphi)\vec{n} + (\cos\theta\sin\varphi)\vec{b} \\
\vec{u}_\psi &= (\sin\varphi)\vec{n} + (\cos\varphi)\vec{b} \\
\vec{u}_\psi &= \vec{t}
\end{aligned} \tag{3}$$

and the corresponding angular velocity of (TdF) is writen as:

$$\vec{\omega}_f = \dot{\psi}_f\vec{u}_\psi + \dot{\theta}_f\vec{u}_\theta + \dot{\phi}_f\vec{u}_\phi \tag{4_1}$$

On the other hand, there is known the expression of this angular velocity as function of the curve curvatures:

$$\vec{\omega}_f = vk_t\vec{t} + vk_n\vec{b} \tag{4_2}$$

Comparing these two equations we derive the ordinary differential system for the angular coordinates of (TdF):

$$\begin{aligned} \dot{\psi}_f \cos \theta_f &= v k_n \sin \varphi_f \\ \dot{\theta}_f \qquad &= v k_n \cos \varphi_f \\ \dot{\phi}_f \cos \theta_f &= v k_t \cos \theta_f + v k_n \sin \theta_f \sin \varphi_f \end{aligned} \tag{5}$$

The next step is to consider the dynamic way to generate the curve, that is the trajectory of a moving point (in fact a punctiform body), to derive the differential system of the cartesian coordinates.

So, the velocity of the moving poit is:

$$\vec{v} = v\vec{t} \tag{6}$$

with:

$$\vec{t} = (\sin \psi_f \cos \theta_f)\vec{I} + (\cos \psi_f \cos \theta_f)\vec{J} + (\sin \theta_f)\vec{K}$$

in the (TdF) frame; and in the (TdO) frame:

$$\vec{v} = \dot{X}\vec{I} + \dot{Y}\vec{J} + \dot{Z}\vec{K} \tag{7}$$

Comparing (6) with (7) we arrive at the desired differential system for the cartesian coordinates X, Y, Z:

$$\dot{X} = v \sin \psi_f \, cos\, \theta_f \quad \dot{Y} = v \cos \psi_f \cos \theta_f \quad \dot{Z} = v \sin \theta_f \tag{8}$$

Collecting the systems (5) and (8) and expressing them as a natural parametric system, that is having s as free variable on, we obtain the differential system for the cartesian coordinates of an intrinsically defined curve, by tacking also into account that d/dt≡vd/ds:

$$\begin{aligned} \psi_{fs} \cos \theta_f &= k_n \sin \varphi_f \\ \theta_{fs} \qquad &= k_n \cos \varphi_f \\ \phi_{fs} \cos \theta_f &= k_t \cos \theta_f + k_n \sin \theta_f \sin \varphi_f \\ X_s &= \sin \psi_f \cos \theta_f \\ Y_s &= \cos \psi_f \cos \theta_f \\ Z_s &= v \sin \theta_f \end{aligned} \tag{9}$$

where the subscript s means the derivative with respect to s.

This differential system can be used to derive the Cartesian coordinates of the curve by numerical methods, when its main curvatures are known as functions of the arc length s, following the definition in (1). To have a validation, we considered a curve derived from an ellipse located in the plane (OXY), which rises along the axis (OZ) each point of it in an oscillating manner, so that it has a length of the period that exactly divides the length of the ellipse; and then the curve thus obtained is rotated. For such a curve there was determined the principal curvatures, k_n and k_t, and then it was recreated the curve using the differential system (9). The result is presented in the diagram below:

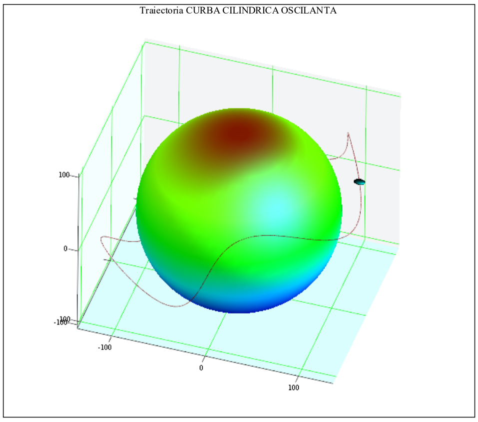

Fig. 3 – The curve generated by the (9) system, positioned around a sfere

3. GENERAL INTEGRAL OF THE DYNAMIC SYSTEM

To build a solution of the dynamic system (9) we notice before that the subsystem for the angular coordinates ψ, θ, φ may be separated:

$$\begin{aligned} \psi_{fs} \cos\theta_f &= k_n \sin\varphi_f \\ \theta_{fs} &= k_n \cos\varphi_f \\ \phi_{fs} \cos\theta_f &= k_t \cos\theta_f + k_n \sin\theta_f \sin\varphi_f \end{aligned} \tag{10}$$

If its possible to derive the values of the coordinate angles as functions of the arc length s, then the coordinates can be found as simple quadratures from the remaining three equations:

$$X = \int_\Gamma \sin\psi \cos\theta \, ds \;\; Y = \int_\Gamma \cos\psi \cos d\, s \;\; Z = \int_\Gamma \sin\theta \, ds \tag{11}$$

The analysis of the differential system (10) becomes necessary to state that the system (9) is integrable in finite terms. We note that from the system (10) it may be possible to separate an independent subsystem as:

$$\begin{aligned} \theta_{fs} &= k_n(s) \cos\varphi_f \\ \phi_{fs} \cos\theta_f &= k_t(s) \cos\theta_f + k_n(s) \sin\theta_f \sin\varphi_f \end{aligned} \tag{12}$$

This differential system may be rewritten as a Pfaff form:

$$0 = (\tan\theta \tan\varphi) d\theta - d\varphi + k_t(s) ds \tag{13}$$

The necessary and sufficient condition for the existence of a general integral of this Paff form, deduced directly from the previous equation, according to the integrability theorem of Frobenius [12], for a three dimensional form, is as follows [10]:

$$(\frac{\partial a}{\partial \varphi} - \frac{\partial b}{\partial \theta})(\lambda c) + (\frac{\partial b}{\partial s} - \frac{\partial c}{\partial \varphi})(\lambda a) + (\frac{\partial c}{\partial \theta} - \frac{\partial a}{\partial s})(\lambda b) = 0 \tag{14}$$

where we have denoted by:

$$a = \tan\theta \tan\varphi \qquad b = -1 \qquad c = k_t(s) \tag{15}$$

and λ is an integrative factor. Doing all the calculations we arrive finally at the result:

$$\lambda k_t(s)\frac{\tan\theta}{\cos^2\varphi}=0 \tag{16}$$

We may consider one of the following consequences. So, for the general case when the principal curvatures are defined independently, there are the following situations:

λ=0	that means no integrative factor appropriate
$k_t(s)$=0	that means curve with no torsion, that is plane curve
$\tan\theta$=0	that means a curve as a traight line

For some particular curvatures it is possible to derive a general integral of the system (12). We will consider as a particular curve the one that fulfills the relationship:

$$\frac{k_t}{k_n}=const(s)\overset{not}{=}K$$

and so we will arrive at the ordinary differential equation:

$$\frac{d\varphi}{d\theta}=\tan\varphi\tan\theta+K\frac{1}{\cos\phi} \tag{17}$$

The solution to this equation is the following one:

$$\sin\varphi\cos\theta=\int K\cos\theta\,d\theta+C_\theta \tag{18}$$

$$\sin\varphi\cos\theta=K\sin\theta+C_\theta$$

where C_θ is an appropriate invariable value. Considering for C_θ the formula for the starting stage of motion we infer:

$$\begin{gathered}\sin\varphi\cos\theta=K\sin\theta+(\sin\varphi_0\cos\theta_0-K\sin\theta_0)\\ \sin\varphi\cos\theta-\sin\varphi_0\cos\theta_0=K(\sin\theta-\sin\theta_0)\end{gathered} \tag{19$_1$}$$

or:

$$\sin\varphi\cos\theta-K\sin\theta=\sin\varphi_0\cos\theta_0-K\sin\theta_0=const. \tag{19$_2$}$$

Now, using the first equation of (12) we derive the following ordinary differential equation:

$$\frac{d\theta}{ds}=\pm k_n\sqrt{1-\left(K\tan\theta+\frac{C_\theta}{\cos\theta}\right)^2} \tag{20$_1$}$$

and

$$\begin{gathered}\frac{d\psi}{ds}=k_n\frac{K\sin\theta+C_\theta}{\cos^2\theta}\\ \frac{d\varphi}{ds}=k_n\frac{K+C_\theta\sin\theta}{\cos^2\theta}\end{gathered} \tag{20$_2$}$$

which may be solved to have the angular coordinate θ as function of free variable s. To do so, we separate the variables in equation (20$_1$), considering that θ is a growing variable:

$$\frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta-(K\sin\theta+C_\theta)^2}}=k_n ds$$

After some algebraic processing, we obtain a more meaningful expression for the radical, bearing in mind that the new variable will be $\sin\theta \overset{not}{=} u$:

$$\sqrt{1-\sin^2\theta-(K\sin\theta+C_\theta)^2}=\sqrt{1-u^2-\left(K^2u^2+2KC_\theta u+C_\theta^2\right)}$$
$$=\sqrt{c-2bu-au^2}=\sqrt{p^2-w^2}$$

in which we have noted:

$$c=1-C_\theta^2\ b=2KC_\theta\ a=1+K^2$$
$$p^2=\frac{ca+b^2}{a^2}w^2=(u+\frac{b}{a})^2 \tag{21}$$

So, the equation (20_1) becomes:

$$\frac{dw}{\sqrt{p^2-w^2}}=\sqrt{a}k_n ds \tag{22}$$

Because it is obviously that $p^2\geq w^2$ the following substitution is possible $w=p\sin z$. The solution of the equation (22) is now, without considering the sign:

$$z=\sqrt{a}\int k_n ds$$
$$\sin\theta=p\sin\left(\sqrt{a}\int k_n ds\right)-\frac{b}{a} \tag{22}$$

and the solution of the equations (20_2) are now simple quadratures.

4. CONCLUSIONS

1. The intuitive meaning of deriving the cartesian coordinates of a 3D curve is based on the trajectory generation by a moving punctiform body having its velocity expressed in (TdF) and (TdO) axes frame.
2. The only curves that admit the expression of their Cartesian coordinates in finite terms are plane or rectilinear curves.
3. For the particular case of k_t/k_n= const(s), (perhaps the only one) one can derive the cartesian coordinates expressed as functions of the free coordinate s.

5. APPENDIX

a. Detailed calculations for the Frobenius integrability theorem applied to the Pfaff form (13), considering the mentioned notations:

$$a=\tan\theta\tan\varphi \qquad b=-1 \qquad c=k_t(s)$$

are as follows:

$$\left(\frac{\partial a}{\partial\phi}-\frac{\partial b}{\partial\theta}\right)(\lambda c)=\left(\tan\theta\frac{\partial\lambda\tan\varphi}{\partial\varphi}-\frac{\partial(-1)\lambda}{\partial\theta}\right)(\lambda k_t)=$$
$$(\lambda\tan\theta\frac{\partial\tan\varphi}{\partial\phi}-\lambda\frac{\partial(-1)}{\partial\theta})(\lambda k_t)+(\tan\theta\tan\varphi\frac{\partial\lambda}{\partial\varphi}-(-1)\frac{\partial\lambda}{\partial\theta})(\lambda k_t)$$

$$(\frac{\partial a}{\partial \varphi}-\frac{\partial b}{\partial \theta})(\lambda c)=(\lambda\frac{\tan\theta}{\cos^2\varphi})(\lambda k_t)+(\tan\theta\tan\phi\frac{\partial\lambda}{\partial\varphi}+\frac{\partial\lambda}{\partial\theta})(\lambda k_t) \tag{23}$$

$$\left(\frac{\partial b}{\partial s}-\frac{\partial c}{\partial \varphi}\right)(\lambda a)=\left(\frac{\partial\lambda(-1)}{\partial s}-\frac{\partial\lambda k_t}{\partial\varphi}\right)(\lambda\tan\theta\tan\varphi)=$$
$$((-1)\frac{\partial\lambda}{\partial s}-k_t\frac{\partial\lambda}{\partial\varphi})(\lambda\tan\theta\tan\varphi)+(\lambda\frac{\partial(-1)}{\partial s}-\lambda\frac{\partial k_t}{\partial\varphi})(\lambda\tan\theta\tan\varphi)$$

$$(\frac{\partial b}{\partial s}-\frac{\partial c}{\partial \varphi})(\lambda a)=(-\frac{\partial\lambda}{\partial s}-k_t\frac{\partial\lambda}{\partial\varphi})(\lambda\tan\theta\tan\varphi) \tag{24}$$

$$\left(\frac{\partial c}{\partial \theta}-\frac{\partial a}{\partial s}\right)(\lambda b)=\left(\frac{\partial\lambda k_t}{\partial\theta}-\frac{\partial\lambda\tan\theta\tan\varphi}{\partial s}\right)\big(\lambda(-1)\big)=$$
$$-(\lambda\frac{\partial k_t}{\partial\theta}-\lambda\frac{\partial\tan\theta\tan\varphi}{\partial s})(\lambda)-(k_t\frac{\partial\lambda}{\partial\theta}-\tan\theta\tan\varphi\frac{\partial\lambda}{\partial s})(\lambda)$$

$$(\frac{\partial c}{\partial \theta}-\frac{\partial a}{\partial s})(\lambda b)=-(k_t\frac{\partial\lambda}{\partial\theta}-\tan\theta\tan\varphi\frac{\partial\lambda}{\partial s})(\lambda) \tag{25}$$

Summing the formulas (23), (24), (25) we remark that all terms cancel each other out and it is derived the final result:

$$\lambda k_t(s)\frac{\tan\theta}{\cos^2\varphi}=0 \tag{26}$$

b. Solving the differential equation (17) we proceed to consider the new unknown function as $\sin\varphi$ which will be denoted by *u*. So, the equation becomes successively:

$$\frac{du}{d\theta}=u\tan\theta+K \quad\rightarrow\quad \frac{du}{d\theta}\cos\theta-u\,sin\,\theta=K\cos\theta$$

It comes to be obvious that the new unknown function is $u\cos\theta$ which will be denoted as *w* and the equation becomes:

$$\frac{dw}{d\theta}=K\cos\theta \tag{27}$$

that has the solution:

$$w=K\sin\theta+C_\theta \tag{28}$$

that is, in fact:

$$\sin\varphi\cos\theta=K\sin\theta+C_\theta \tag{29}$$

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