Study of the railway vehicle suspension using the multibody method

Ioan SEBESAN*[1], Dan BAIASU[2], Gheorghe GHITA[3]

*Corresponding author
*[1] Faculty of Transports, "POLITEHNICA" University of Bucharest,
Splaiul Independentei 313, 060042, Bucharest, Romania
ioan_sebesan@yahoo.com
[2]Atelierele CFR Grivita S.A.,
Calea Grivitei 359, 010178, Bucharest, Romania
dan.baiasu@grivita.ro
[3] IMS-AR - Institute of Solid Mechanics of the Romanian Academy,
C-tin Mille 15, 10141, Bucharest, Romania
gh_ghita@yahoo.com

DOI: 10.13111/2066-8201.2011.3.4.9

Abstract: *The article presents a mathematical model for the study of a passenger coach hunting motion using the multibody approach. The model comprises the lateral displacement, rolling and yawing motions for the main constitutive elements: axles, bogies and case. The equation system is written applying energetic methods. The forced vibrations determined by the irregular profile of the tracks are considered. The wheel – rail contact forces are expressed using the creepage coefficients established according to Kalker's linear theory. The equations system is solved through numeric methods using specialized calculus programs. The response of the system – passenger coach on a tangent track, the critical speed and the influence of the constructive characteristics of the coach on its performances are determined.*

Key Words: *multibody system method; mathematical model; mechanical system response; system lateral stability; suspension study*

1. INTRODUCTION

The lateral railway vehicle dynamics represents a field of great interest in the actual context where more and more railway administrations implement the high speed trains, which prove to be efficient, economic and ecological transportation means. Trains circulating with speeds higher than 160 km/h generate vibrations in the vehicle body that may induce significant operation problems: running instability, poor ride quality and track wear. From this point of view, an adequate design of the railway vehicles' suspensions holds an important role in maintaining the comfort and safety parameters of trains circulation.

Kinematic theoretical studies of the rolling apparatus' elements lateral and yawing motions [1], [2] have highlighted that the oscillation frequency grows proportionally to the circulation speed. The speed value where the amplitude of the oscillations grows and the vehicle movement becomes unstable is called critical speed. Starting off with this approach, various studies on the railway vehicle's lateral stability have showed the existence of two sources of instability for the railway vehicle:

- the bogie instability, induced by the axles movement instability;

- the instability of the case, which appears when, in the low frequency domain, the vehicle body has the tendency of moving along with the bogie.

The dynamic behavioral study of the railway vehicle has two directions:

- the dynamic response of the system: simulating dynamic behavior due to external stimuli, determining the concentrated mass accelerations and speeds, and implicitly the forces that act upon the vehicle;
- the dynamic stability: the study over the system's stability in various operation conditions.

Starting with the 60's, numerous authors have dedicated studies to the lateral oscillations phenomenon (the hunting motion): Wickens (1965), Law and Coperrider (1974), Garg and Dukkipatti (1984), Sebesan (1995), Ahmadian and Yang (1998), He and McPhee (2002), Fan and Wu (2006), Messouci (2009), Wang and Liao (2009) and others.

The mathematical models used in the literature for the study of vehicles, differ depending on the number of degrees of freedom taken into account, the vehicle type, the linear or non-linear treatment of the wheel – rail contact phenomenon, of the forces appearing at the wheel – rail contact, as well as the irregularities of the tracks. The complexity of those models has evolved in the same time as the computing technique has become more performant allowing to find solutions for more and more complex sets of differential equations using up to 38 degrees of freedom and taking into consideration more and more non-linear aspects of the vehicle – rail interaction. Numerous mechanical models built until now – [4], [5], [6], [11], [12] concerning especially the oscillations of the wheelset, considering that these determine the vibration regime in the whole vehicle, are of interest because they allow the study of the non-linearities specific to the processes generated by the rolling of the mounted axle on the tracks or the assessment of the importance of various constructive parameters of the vehicle, but cannot represent the phenomena that take place at the level of the case – bogies connection. Moreover, few of the mechanical models presented in the literature are validated through dynamic tests [8], [9], [10], [13].

This article presents a mathematical model of a passenger coach established using the multibody system method presented in [14] to simulate the response from the oscillating system to the irregularities of the tracks and the critical speed of the coach. Simulating the vehicle's response for various values of its constructive parameters facilitates the study of optimization possibilities for the coach's performance.

2. METHOD FORMULATION

Multibody method is an analytical dynamics approach that considers the mechanical connections and the trajectories that impose constraints to the motion and not the external forces. Constraints equations reduce the number of degrees of freedom in comparison to the methods treating the external forces. The vehicle is considered as it is composed of a limited number of rigid bodies, simulating its main parts, connected in between through mechanical weightless linkages. A rigid body is identified using 6 coordinates. For a railway vehicle those coordinates are defining the following movements:

- the translations:
    - x – translation along the rail;
    - y – lateral oscillation;
    - z - bouncing;
- the orientations:

- ϕ - rolling;
- ψ - yawing;
- θ - pitching.

Inertial reference frames coordinate system whose origin is fixed in space and time presented in [14] has been extended in this article to simulate the lateral dynamics of a passenger coach.

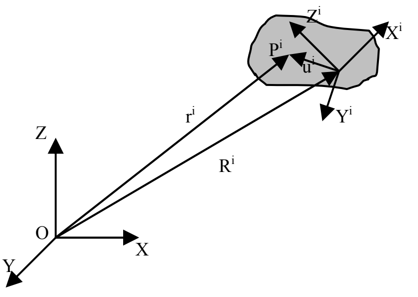

Fig. 1 Coordinate systems of a rigid body

Considering a rigid body i in a three-dimensional space, the global position of an arbitrary point P^i is expressed as being:

$$r^i = R^i + A^i\overline{u}^i \tag{1}$$

$R^i = \left[R_1^i \ R_2^i \ R_3^i\right]$ is the global position vector of the origin of the body reference frame, A^i is the transformation matrix from the body coordinate system to the global coordinate system, $\overline{u}^i = \left[u_1^i \ u_2^i \ u_3^i\right]$ is the position vector of the point P^i with reference to the body coordinate system. To express the transformation matrix there are commonly used four methods: the Euler angles, the Euler parameters, the Rodriguez parameters and the direction cosines. In this mathematical model it will be used the Euler angles and the transformation matrix will be:

$$A^i = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\varphi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\varphi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \tag{2}$$

Using the equation (1) there can be obtained the velocity and acceleration vectors:

$$\begin{gathered} \dot{r}^i = \dot{R}^i + \dot{A}^i\overline{u}^i = \dot{R}^i + \omega^i \times u^i = \dot{R}^i + A^i(\overline{\omega}^i \times \overline{u}^i) \\ \ddot{r}^i = \ddot{R}^i + \ddot{A}^i\overline{u}^i = \ddot{R}^i + \omega^i \times (\omega^i \times u^i) + \alpha^i \times u^i = \ddot{R}^i + A^i[\overline{\omega}^i \times (\overline{\omega}^i \times \overline{u}^i)] + A^i(\overline{\alpha}^i \times \overline{u}^i) \end{gathered} \tag{3}$$

where $\overline{\omega}^i$ and $\overline{\alpha}^i$ are, respectively, the angular velocity vector and angular acceleration vector defined in the body coordinate system. The angular speed of the body using the Euler angles can be written:

$$\begin{gathered} \omega = G\dot{\upsilon} \\ \upsilon = (\phi, \theta, \psi) \\ G = \begin{bmatrix} 0 & \cos\phi & \sin\theta\sin\phi \\ 0 & \sin\phi & -\sin\theta\cos\phi \\ 1 & 0 & \cos\theta \end{bmatrix} \end{gathered} \tag{4}$$

The angular speed vector is:

$$\overline{\omega} = \overline{G}\dot{\upsilon} \tag{5}$$

$$\overline{G} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}$$

The method reveals several advantages, one being the fact that the specific railway linkages can be easily simulated: linear spring and damper connection, linear spring and non-linear damper, contact between rigid or deformable bodies, articulation type connection. The constraint conditions are expressed as nonlinear algebraic equations, depending on the generalized coordinates and time that leads to a system of algebraic and differential equations. There are two formulation methods for the force-acceleration equations:

- the augmented formulation: the constraints equations are added to the system differential equations of motion leading to a system that includes the constraint forces;
- the embedded formulation: the constraint forces are used to systematically eliminate the dependent coordinate and the constraint forces, leading to a reduced system of differential equations associated with the degrees of freedom of the system.

For the linear spring and damper connection between two bodies i and j at points P^i and P^j, the expression of the force acting in the suspension element between the bodies is:

$$F = k(l - l_0) + c\dot{l} \qquad (6)$$

where k is the spring constant, c the damping coefficient, l_0 the undeformed length of the spring and l the length of the loaded spring.

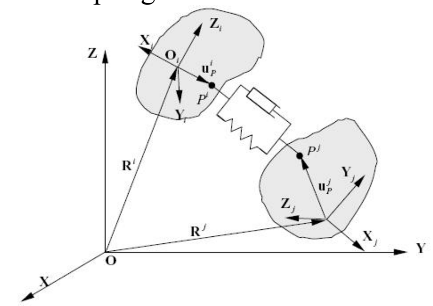

Fig. 2 Linear spring and damper connection

If r_P^{ij} is the position vector of point P^i with respect to P^j:

$$r_P^{ij} = r_P^i - r_P^j = R^i + A^i\overline{u}_P^i - R^j - A^j\overline{u}_P^j \qquad (7)$$

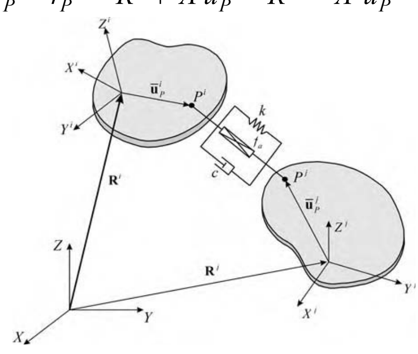

Fig. 3 Semi-active connection

For a non-linear connection containing a semi-active magneto rheological damper described by the Bouc-Wen model, the force is given by the following expression:

$$F = c\dot{l} + k(l - l_0) + \alpha z, \qquad \dot{z} = -\gamma|\dot{x}|z|z|^{n-1} - \beta\dot{x}|z|^{n} + \delta\dot{x} \tag{8}$$

where α , β , γ , n are specific parameters of the model.

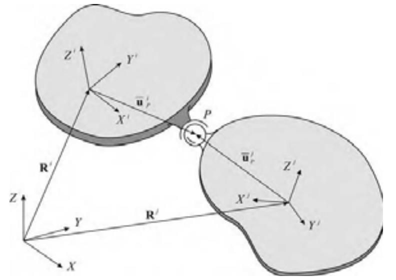

Fig. 4 Spherical joint connection

The spherical joint connection allows only the rotational movements of the two bodies and eliminates the translations. Its equation is:

$$r_P^i - r_P^j = 0 \tag{9}$$

Another type of joint widely used on the railway coaches is the revolute joint.

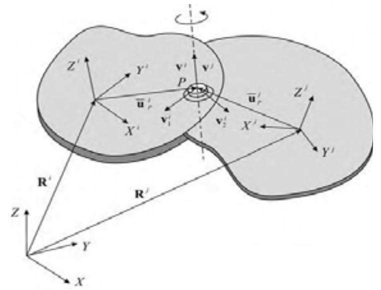

Fig. 5 Revolute joint connection

The revolute joint connection allows only one relative rotation along the joint axis and eliminates the rest of five degrees of freedom. Imposing the parallelism condition of the vectors that represents the rigid bodies, the constraint equation of this type of connection is:

$$\begin{bmatrix} r_P^i - r_P^j \\ v_1^i \cdot v^j \\ v_2^i \cdot v^j \end{bmatrix} = 0 \tag{10}$$

For a contact connection between two bodies there can be defined in the contact point, for each of the bodies, two vectors along two tangent orthogonal directions.

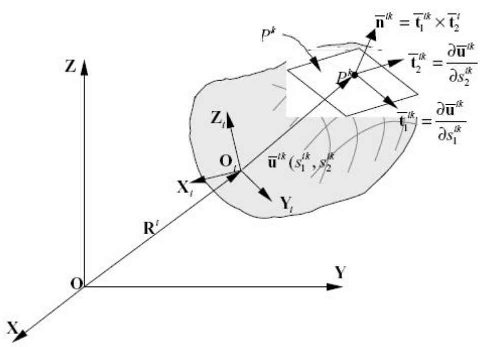

Fig. 6 Contact connection

$$t_{1,2}^{ik} = \frac{\delta \overline{u}^{ik}}{\delta s_{1,2}^{ik}}, \qquad t_{1,2}^{jk} = \frac{\delta \overline{u}^{jk}}{\delta s_{1,2}^{jk}} \tag{11}$$

The vectors previously generated define for each surface in contact a normal vector in the contact point:

$$\overline{n}^{ik} = t_1^{ik} \times t_2^{ik}, \qquad \overline{n}^{jk} = t_1^{jk} \times t_2^{jk} \tag{12}$$

From the condition of contact between the surfaces – same tangent plan in the contact point – results that the two normal vectors coincides. The constraint equation is:

$$R^i + A^i \overline{u}^{ik} - R^j - A^j \overline{u}^{jk} = 0 \qquad n^{jkT} \cdot t_{1,2}^{jk} = 0 \tag{13}$$

If the body is considered elastically deformable, the global position of an arbitrary point P^i is expressed as being:

$$\overline{u}^i = \overline{u}_0^i + \overline{u}_f^i \qquad r^i = R^i + \overline{u}_0^i + \overline{u}_f^i \tag{14}$$

where $\overline{u}_0^i$ is the undeformed local position of point P^i, and $\overline{u}_f^i$ is the deformation vector at P^i.

The speed and acceleration of the body are:

$$\begin{aligned} \dot{r}^i &= \dot{R}^i + \omega^i \times (u_0^i + u_f^i) + (\dot{u}_f^i)_r \\ \ddot{r}^i &= \ddot{R}^i + \omega^i \times (\omega^i \times u^i) + \alpha^i \times u^i + 2\omega^i \times (\dot{u}_f^i)_r + \left(\ddot{u}_f^i\right)_r \end{aligned} \tag{15}$$

The generalized coordinates of the body reference can be written as:

$$q_r^i = \left[R^{i^T} \ \ \Theta^{i^T}\right]^T, \tag{16}$$

where R^i the position coordinates vector, Θ^i the orientation coordinates vector.

If considering that the origin of the rigid body reference is positioned in the mass center of the body, the mass matrix of the rigid body i can be written as:

$$M^i = \begin{bmatrix} m_R^i & 0 \\ 0 & m_\Theta^i \end{bmatrix} \tag{17}$$

The kinetic energy of the rigid body i:

$$T^i = \frac{1}{2}\left[\dot{R}^{i^T} \ \ \dot{\Theta}^{i^T}\right] \cdot \begin{bmatrix} m_R^i & 0 \\ 0 & m_\Theta^i \end{bmatrix} \cdot \begin{bmatrix} \dot{R}^i \\ \dot{\Theta}^i \end{bmatrix} \tag{18}$$

The virtual work of the eternal forces partitioned with respect to translation and rotation of the body reference:

$$\delta W^i = [(Q_R^i)^T \ \ (Q_\Theta^i)^T] \begin{bmatrix} \delta R^i \\ \delta \Theta^i \end{bmatrix} \tag{19}$$

The kinematical constraints between different components of the multibody system can be written in a vector form that describes mechanical connections in the system as well as specified motion trajectories, as $C(q,t) = 0$, where C is the vector of linearly independent constraint equations, t is time, and q is the total vector of the multibody system generalized coordinates. The system equations of motion of the rigid body i in the multibody system using an energetic method can be written as:

$$M^i \ddot{q}_r^i + C_{q_r^i}^T \lambda = Q_e^i + Q_v^i \tag{20}$$

where M^i is the mass matrix, $C_{q_r^i}^T$ is the constraint Jacobian matrix, λ is the vector of Lagrange multipliers, Q_e^i is the vector of externally applied forces, and Q_v^i is a quadratic velocity vector that arises from differentiating the kinetic energy with respect to time and to the generalized coordinates of body i . Considering $i = 1,2,..., n_b$ in equation (12) results the multibody system equations of motion which can be expressed in a matricial form as:

$$M\ddot{q} + C_q^T \lambda = Q_e + Q_v \quad (21)$$

3. SIMULATION OF THE LATERAL DYNAMICS OF THE RAILWAY VEHICLE

The specific construction characteristics of the vehicle were considered in order to elaborate the model. The dynamic response of the coach on bogies at the tracks' irregularities on a horizontal plan depends strongly on the configuration of the tracks which represent inputs for the vehicle as an oscillating system, and also depends on its circulation speed. An accurate design of the axle suspension can reduce the high frequency vibrations generated by the tracks, by the reduced conicity of the rolling tread and by the wheel – rail contact forces. The central suspension must absorb the vibrations transmitted by the bogies to the coach case in order to maintain the vehicle in an optimal position during circulation and so that it ensures the passengers comfort. Both suspensions also contain dissipative elements that attenuate the vibrations generated by the vehicle's movement. Usually, for railway vehicles, the mathematical models consider the body case, the bogies and the wheelsets as concentrated mass. The equations system describing the movement of the mathematical model could thus have 42 quadratic coupled equations. Solving such a system represents a sometimes inconclusive undertaking regarding the vehicle's behavior. According to [1], [3], [4] for small amplitude movements, there is a relatively small connection between the vehicle's oscillations on a vertical and transversal directions, this is why some of the models presented in literature don't take into account the vertical oscillations in the study of movement on lateral direction or the horizontal oscillations for the study of vehicle vertical displacement. A reliable model of the vehicle must include both suspension levels, allowing the highlighting of the dynamic characteristics of the vehicle's movement and the study of displacements appearing between the components of the model. In the process of building the model, the coach case, the bogies' frames and the axles have been considered as rigid concentrated masses. The coach case center of mass is located at the h_{cc} height from the separation plan between the case and the bogie's frame – the transversal plan that equally divides the bogie's central suspension's coil spring. This plan is located at an h_{cb} distance from the bogie's mass center.

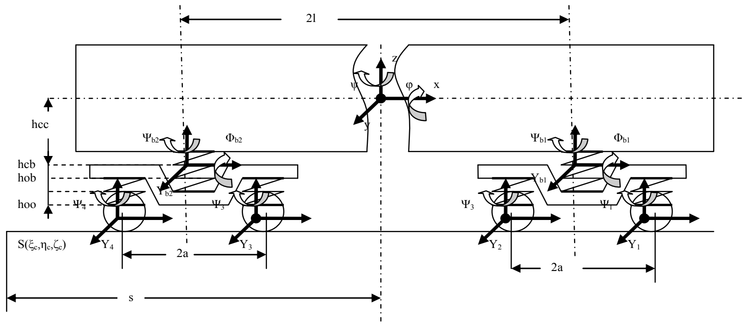

Fig. 7 Mechanical model for passenger coach's hunting (lateral view)

It is assumed that all the elastic and damping elements are weightless and have linear characteristics: the elastic force is directly proportional with the coil spring deformation and the damping force is directly proportional with the dampers' deformation speed. In order to deduce the mechanical model's movement equations a preliminary establishment of the reference systems and coordinates describing the movement of the concentrated mass inside the model was necessary. The multibody model contains the following elements:

- the coach case;
- the bogies b_j , j=1,2;
- the wheelsets o_i , i=1...4;
- O_c, O_{bj}, O_i – the centers of mass for the mechanical model elements;
- x_c, y_c, z_c, ψ_c, φ_c, θ_c – the translations, respectively rotations, of the coach case during movement;
- x_{bj}, y_{bj}, z_{bj}, ψ_{bj}, φ_{bj}, θ_{bj} – the displacements, respectively rotations, of the bogies;
- x_{oi}, y_{oi}, z_{oi}, ψ_{oi}, φ_{oi}, θ_{oi} – the displacements, respectively rotations, of the mounted axles;
- h_c – distance between O_c and O_{bj} ;
- h_b – distance between O_{bj} and O_i .

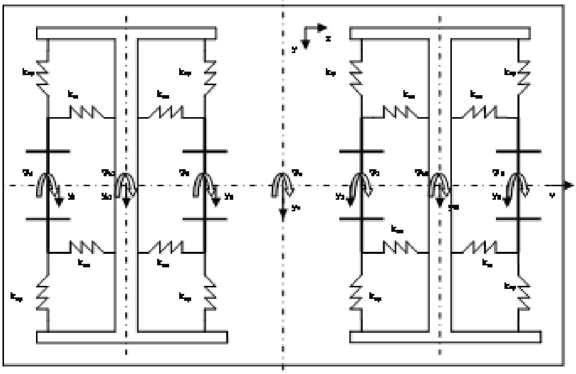

Fig. 8 Mechanical model for passenger coach’s hunting (upper view)

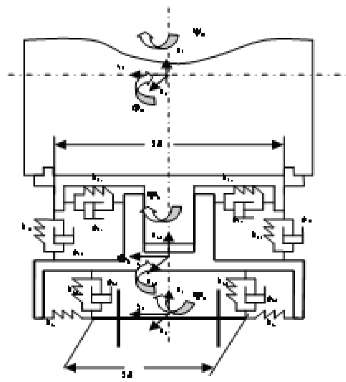

Fig. 9 Mechanical model for passenger coach’s hunting (transversal view)

Considering the coach as a system of rigid bodies interconnected through suspension elements, under conditions of geometrical, elastic and inertial symmetry, with identical wheel and rail patterns, the equilibrium position of the coach coincides with its median position in relation to the tracks. The yawing motions of the coach around its equilibrium position were considered to be of relatively small amplitudes, without moving in all the available space in the vehicle slot guide. In this case, the rolling surfaces' contact angles are small, the radii of curvature for the rolling treads remain unchanged and the expression for the centering gravitational force can be linearized. Conicity has been considered as having an equal constant value with the rolling surfaces' effective conicity. The small contact angles create the premises for neglecting the contact forces' vertical components in relation with the wheel loads which can be considered equal to the normal contact forces. Adopting the hypothesis of the small oscillations implies the existence of transversal accelerations in the small amplitude vehicle which signifies that the load transfers between the wheels of the same axle can be neglected. At the same time, the axle's vertical accelerations at small frequencies characteristic to yawing can be neglected, the axle load can be considered as being constant so that the wheel load is also considered as being constant. The mechanical model's geometrical and elastic symmetry facilitates the decoupling of the lateral movements from the vertical ones. In order to study the 4 axles vehicle's lateral oscillations, considering that we are using the simplifying hypotheses previously presented, the mechanical model considers the following degrees of freedom: y_c, ψ_c, φ_c, y_{bj}, ψ_{bj}, φ_{bj},y_i, ψ_i, where j=1,2 represent the bogies and i=1 – 4 the wheelsets.

Hence, a system results, with 17 degrees of freedom corresponding to the concentrated mass movements. According to Kalker's linear theory, both the creep tangential forces T_x and T_y and M_z the creep moment act in the contact point wheel – rail:

$$\begin{aligned} T_x &= \chi_x \nu_x Q \\ T_y &= \chi_y \nu_y Q + \chi_s r_0 (\omega_s / v) Q \\ M_z &= -\chi_s r_0 \nu_y Q + \chi_z r_0^2 (\omega_s / v) Q \end{aligned} \tag{22}$$

where the spin creepage is given by the expressions:

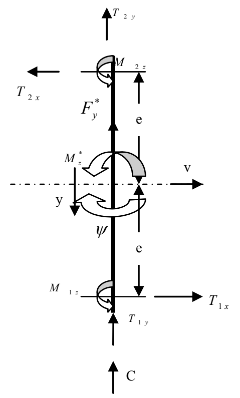

Fig.10 Forces and moment acting on the wheelset

$\omega_{se} = \omega_y \sin \gamma_e$ - corresponding to wheel 1

$\omega_{si} = \omega_y \sin \gamma_i$ -corresponding to wheel 2, where $\omega_y = v / r$ represents the angular speed transversal component in the wheel – rail contact point.

According to [1], approximate values are indicated for the creepage coefficients: $\chi_x \approx \chi_y = \chi = \dfrac{300}{\sqrt[3]{Q}} ... \dfrac{400}{\sqrt[3]{Q}}$ (for Q expressed in tons), which depend on the ratio of the contact ellipse axes.

For the spin coefficient χ_s, the literature recommends a value of 0,83 because it is almost independent in regard to the ratio of the contact ellipse axes. The χ_z coefficient for a circular contact surface is $\chi_z = 0{,}0043\sqrt[3]{Q}$ and for a contact surface whose axis length in the running direction is twice, respectively 0,5 times the length of the other axis is $\chi_z = 0{,}0014\sqrt[3]{Q}$ and, respectively, $\chi_z = 0{,}0134\sqrt[3]{Q}$ .The χ_z coefficient has a reduced influence over the yawing motion and can be neglected.

The creepages in the contact points of the two axle wheels have the expressions:

$$\begin{aligned} \nu_{1x} &= -\nu_{2x} = -[(\gamma / r_0) y + (e/v)\dot{\psi}] \\ \nu_{1y} &= \nu_{2y} = \dot{y}/v - \psi \\ \omega_{1s} &= -(v/r_0)\gamma_1 \\ \omega_{2s} &= (v/r_0)\gamma_2 \end{aligned} \tag{23}$$

In the contact points, the forces and the moments will have expressions given by the following relations:

$$T_{1x} = -\chi Q(\frac{\gamma}{r_0}y + \frac{e}{v}\dot{\psi}) \qquad T_{2x} = \chi Q(\frac{\gamma}{r_0}y + \frac{e}{v}\dot{\psi})$$

$$T_{1y} = \chi Q(\frac{\dot{y}}{v} - \psi) - \chi_s Q\gamma_1 \qquad T_{2y} = \chi Q(\frac{\dot{y}}{v} - \psi) + \chi_s Q\gamma_2 \tag{24}$$

$$M_{1z} = M_{2z} = -\chi_s Qr(\frac{\dot{y}}{v} - \psi)$$

The centering force:

$$C = Q(\gamma_1 - \gamma_2) = c_g y \tag{25}$$

An inertial reference frame is considered – $O\xi\eta\zeta$ originating in the wheelset plan, on the tracks axis, at a distance s from the O_c coach's center of mass (fig. 3). In order to determine the relative displacements of the multibody model's elements it will be used the (7) formulae presented previously. The position vector of the center of the central suspension with respect to the case is:

$$r^c = R^c + A^c\overline{u}^c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} + \begin{bmatrix} 1 & -\psi_c & 0 \\ \psi_c & 1 & -\phi_c \\ 0 & \phi_c & 1 \end{bmatrix} \cdot \begin{bmatrix} (-1)^j a \\ \pm d_s \\ -h_{cc} \end{bmatrix} \tag{26}$$

The position vector of the center of the central suspension with respect to the bogies is:

$$r^{b_j} = R^{b_j} + A^{b_j}\overline{u}^{b_j} = \begin{bmatrix} x_{b_j} \\ y_{b_j} \\ z_{b_j} \end{bmatrix} + \begin{bmatrix} 1 & -\psi_{b_j} & 0 \\ \psi_{b_j} & 1 & -\phi_{b_j} \\ 0 & \phi_{b_j} & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \pm d_s \\ h_{cb} \end{bmatrix} \tag{27}$$

The vector of the relative displacements between the case and the bogies results applying (7) and considering the geometry of the multibody model $x_c - x_{b_j} = -(-1)^j a$, $z_c - z_{b_j} = h_{cc} + h_{cb}$:

$$r^c - r^{b_j} = \begin{bmatrix} -\left(\psi_c - \psi_{b_j}\right)(\pm d_s) \\ y_c + h_{cc}\varphi_c + (-1)^{j+1} l\psi_c - y_{bj} + h_{cb}\varphi_{bj} \\ (\varphi_c - \varphi_{bj})(\pm d_s) \end{bmatrix} \tag{28}$$

In a similar way it results the relative displacement vector in the axle suspension, between the bogies and the axles:

$$r^{b_j} - r^i = \begin{bmatrix} -(\psi_{bj} - \psi_i)(\pm d_o) \\ y_{bj} + (-1)^{i+1} a\psi_{bj} + h_{ob}\varphi_{bj} - y_i \\ (\pm d_o)\varphi_{bj} \end{bmatrix} \tag{29}$$

Lagrange's equation method may be applied as follows, in order to establish the movement equations:

$$\frac{d}{dt}\left[\frac{\partial(E-V)}{\partial \dot{q}_k}\right] - \frac{\partial(E-V)}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k \tag{30}$$

where, q_k - generalized coordinate, $\dot{q}_k$ - generalized speed, E – kinetic energy, V – potential energy, D – energy dissipation function; Q_k – generalized force corresponding to the generalized coordinate q_k. The oscillating system's kinetic energy, potential energy and the energy dissipation function have the expressions:

$$T=\frac{1}{2}m_c\dot{y}_c^2+\frac{1}{2}I_{cz}\psi_c^2+\frac{1}{2}I_{cx}\varphi_c^2+\frac{1}{2}m_b\sum_{j=1}^{2}\dot{y}_{bj}^2+\frac{1}{2}I_{bz}\sum_{j=1}^{2}\dot{\psi}_{bj}^2+\frac{1}{2}I_{bx}\sum_{j=1}^{2}\varphi_{bj}+\frac{1}{2}m_o\sum_{i=1}^{4}\dot{y}_i^2+\frac{1}{2}I_{oz}\sum_{i=1}^{4}\dot{\psi}_i^2$$

$$\begin{aligned}V=\ &k_{cy}\sum_{j=1}^{2}(y_c+h_{cc}\varphi_c+(-1)^{j+1}l\psi_c-y_{bj}+h_{cb}\varphi_{bj})^2+k_{cx}\sum_{j=1}^{2}\left[(\psi_c-\psi_{bj})(\pm d_c)\right]^2+\\&k_{cz}\sum_{j=1}^{2}\left[(\varphi_c-\varphi_{bj})(\pm d_c)\right]^2+k_{oy}\sum_{j=1}^{2}\sum_{i=1}^{4}(y_{bj}+(-1)^{i+1}a\psi_{bj}-h_{ob}\varphi_{bj}-y_i)^2+k_{ox}\sum_{j=1}^{2}\sum_{i=1}^{4}\left[(\psi_{bj}-\psi_i)(\pm d_o)\right]^2+\\&k_{oz}\sum_{j=1}^{2}\sum_{i=1}^{4}\left[\varphi_{bj}(\pm d_o)\right]^2\\D=\ &\rho_{cy}\sum_{j=1}^{2}(\dot{y}_c+h_{cc}\dot{\varphi}_c+(-1)^{j+1}l\dot{\psi}_c-\dot{y}_{bj}+h_{cb}\dot{\varphi}_{bj})^2+\rho_{cx}\sum_{j=1}^{2}\left[(\dot{\psi}_c-\dot{\psi}_{bj})(\pm d_c)\right]^2+\\&\rho_{cz}\sum_{j=1}^{2}\left[(\dot{\varphi}_c-\dot{\varphi}_{bj})(\pm d_c)\right]^2+\rho_{oz}\sum_{j=1}^{2}\sum_{i=1}^{4}\left[\dot{\varphi}_{bj}(\pm d_o)\right]^2\end{aligned}\tag{31}$$

According to the contact forces (4), the generalized forces corresponding to the generalized coordinates y_i si ψ_i, have the following expressions:

$$\begin{aligned}Q_{yi}&=-2\chi Q\left(\frac{\dot{y}_i}{v}-\psi_i\right)-c_g(1-\chi_s)(y_i-\eta_i)\\Q_{\psi i}&=-2\chi Qe\left[\frac{\gamma}{r_0}(y_i-\eta_i)+\frac{e}{v}\dot{\psi}_i\right]+2\chi_s Qr_0\left(\frac{\dot{y}_i}{v}-\psi_i\right)\end{aligned}\tag{32}$$

where, χ - the creepage coefficient,v – the coach's circulation speed, Q – wheel load, γ - effective conicity of the tread, r_0 – the wheel tread radius, χ_s - spin creepage coefficient, c_g – gravitational stiffness, η_i - track deviations on transversal direction.

The mathematical model considers the aspects of the vibrations introduced by the irregularities of the tracks. According to [1], [4], the expression of the alignment deviations is possible in a sinusoidal form (33):

$\eta_{1,2}=\eta_0\cos[2\pi(vt+l\pm a)/L]$; $\eta_{3,4}=\eta_0\cos[2\pi(vt-l\pm a)/L]$ for the four wheelsets.

Applying Lagrange's equations one can obtain the movement equations for the coach case, bogies and axles:

$$\begin{aligned}&m_c\ddot{y}_c+2k_{cy}\left[2(y_c+h_{cc}\varphi_c)-(y_{b1}+y_{b2})+h_{cb}(\varphi_{b1}+\varphi_{b2})\right]+\\&2\rho_{cy}\left[2(\dot{y}_c+h_{cc}\dot{\varphi}_c)-(\dot{y}_{b1}+\dot{y}_{b2})+h_{cb}(\dot{\varphi}_{b1}+\dot{\varphi}_{b2})\right]=0\\[2ex]&(I_{cz}/2)\ddot{\psi}_c+2(\rho_{cy}l^2+\rho_{cx}d_c^2)\dot{\psi}_c+2(k_{cy}l^2+k_{cx}d_c^2)\psi_c-\rho_{cy}l(\dot{y}_{b1}-\dot{y}_{b2})+\\&\rho_{cy}l(\dot{\varphi}_{b1}-\dot{\varphi}_{b2})h_{cb}-\rho_{cx}d_c^2(\dot{\psi}_{b1}+\dot{\psi}_{b2})-k_{cy}l(y_{b1}-y_{b2})+k_{cy}l(\varphi_{b1}-\varphi_{b2})h_{cb}-\\&k_{cx}d_c^2(\psi_{b1}+\psi_{b2})=0\\[2ex]&(I_{cx}/2)\ddot{\varphi}_c+2(\rho_{cy}h^2{}_{cc}+\rho_{cz}d_c^2)\dot{\varphi}_c+2(k_{cy}h_{cc}^2+k_{cz}d_c^2)\varphi_c+2\rho_{cy}h_{cc}\dot{y}_c-\rho_{cy}h_{cc}(\dot{y}_{b1}+\dot{y}_{b2})+\\&(\rho_{cy}h_{cc}h_{cb}-\rho_{cz}d_c^2)\cdot(\dot{\varphi}_{b1}+\dot{\varphi}_{b2})+2k_{cy}h_{cc}y_c-k_{cy}h_{cc}(y_{b1}+y_{b2})+\\&(k_{cy}h_{cc}h_{cb}-k_{cz}d_c^2)(\varphi_{b1}+\varphi_{b2})=0\\[2ex]&m_b\ddot{y}_{bj}+2\rho_{cy}\dot{y}_{bj}+2(k_{cy}+2k_{oy})y_{bj}-2\rho_{cy}\dot{y}_c-2\rho_{cy}h_{cc}\dot{\varphi}_c-2(-1)^{j+1}\rho_{cy}l\dot{\psi}_c-2(\rho_{cy}h_{cb})\dot{\varphi}_{bj}-\\&2k_{cy}y_c-2k_{cy}h_{cc}\varphi_c-2(-1)^{j+1}k_{cy}l\psi_c-2(k_{cy}h_{cb}+2k_{oy}h_{ob})\varphi_{bj}-2k_{oy}(y_{2j-1}+y_{2j})=0\end{aligned}\tag{33}$$

$$(I_{bz}/2)\ddot{\psi}_{bj}+\rho_{cx}d_c^2\dot{\psi}_{bj}+\left(k_{cx}d_c^2+2k_{oy}a^2+2k_{ox}d_o^2\right)\psi_{bj}-\rho_{cx}d_c^2\dot{\psi}_c-k_{cx}d_c^2\psi_c-k_{oy}a\left(y_{2j-1}-y_{2j}\right)-$$
$$k_{ox}d_o^2\left(\psi_{2j-1}+\psi_{2j}\right)=0$$

$$(I_{bx}/2)\ddot{\varphi}_{bj}+\left(\rho_{cy}h_{cb}^2+\rho_{cz}d_c^2+2\rho_{oz}d_o^2\right)\dot{\varphi}_{bj}+\left(k_{cy}h_{cb}^2+k_{cz}d_c^2+2k_{oy}h_{ob}^2+2k_{oz}d_o^2\right)\varphi_{bj}+\rho_{cy}h_{cb}\dot{y}_c+$$
$$\left(\rho_{cy}h_{cb}h_{cc}-\rho_{cz}d_c^2\right)\dot{\varphi}_c+$$
$$+(-1)^{j+1}l\rho_{cy}h_{cb}\dot{\psi}_c-\rho_{cy}h_{cb}\dot{y}_{bj}+k_{cy}h_{cb}y_c+\left(k_{cy}h_{cb}h_{cc}-k_{cz}d_c^2\right)\varphi_c+(-1)^{j+1}lk_{cy}h_{cb}\psi_c-$$
$$\left(k_{cy}h_{cb}+2k_{0y}h_{ob}\right)y_{bj}++k_{oy}h_{ob}\left(y_{2j-1}+y_{2j}\right)=0$$

$$m_o\ddot{y}_i+2\frac{\chi Q}{v}\dot{y}_i+2\left(k_{oy}+\frac{c_g(1-\chi_s)}{2}\right)y_i-2\chi Q\psi_i-2k_{oy}y_{bj}+2(-1)^{j+1}k_{oy}a\psi_{bj}+$$
$$2k_{oy}h_{ob}\varphi_{bj}=c_g\left(1-\chi_s\right)\eta_i$$

$$I_{oz}\ddot{\psi}_i+\left(2\chi Q\frac{e^2}{v}\right)\dot{\psi}_i+2\left(k_{ox}d_o^2+\chi_s Qr_0\right)\psi_i-\left(2\chi_s\frac{Qr_0}{v}\right)\dot{y}_i+2\chi Q\frac{e\gamma}{r_o}y_i-$$
$$2k_{ox}d_o^2\psi_{bj}=2\chi Q\frac{e\gamma}{r_0}\eta_i$$

4. THE RAILWAY VEHICLE RESPONSE

The general form of the movement equations for the system with more degrees of freedom is:

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{F(t)\} \tag{34}$$

where, $[M]$ – mass matrix, $[C]$ – damping matrix, $[K]$ – stiffness matrix, $\{\ddot{q}\}$ - acceleration vector, $\{\dot{q}\}$ - speed vector, $\{q\}$ - generalized coordinates vector, $\{F(t)\}$ - force vector.
Thus, for the system with 17 degrees of freedom, the displacements vector is:

$$\{q\}=\left[y_c\ \psi_c\ \varphi_c\ y_{b1}\ \psi_{b1}\ \varphi_{b1}\ y_{b2}\ \psi_{b2}\ \varphi_{b2}\ y_1\ \psi_1\ y_2\ \psi_2\ y_3\ \psi_3\ y_4\ \psi_4\right] \tag{35}$$

The mass matrix is a square matrix of order 17, with mass on diagonal and moments of inertia of the concentrated mass composing the mechanical model associated to the coach previously presented. The $[C]$ and $[K]$ matrices are square matrices of order 17 made up of the damping coefficients and the stiffness of the mechanical system. Because it is not possible to establish analytical expressions in relation to the system's response or the critical speed, both the study of movement stability and the determination of the hunting oscillations amplitudes are made using a numerical integration method of the movement equations, the Runge – Kutta method of 4th order, for which the MATLAB program package has specific procedures. Using the constructive data presented in the Table 1 it was performed a numerical simulation in MATLAB. In the simulation it was considered that the coach is launched on a tangent track and runs with a constant speed. In the movement equations' general expressions the coach was considered as an oscillating system activated by the tracks' irregularities. The elements' response was thus established – concentrated masses that make up the coach's mechanical model, translated in the generalized displacements' diagrams in relation to time at the maximum speed at which the coach is checked in the test polygon – 180 km/h, presented in fig. 11 -18. The diagram study indicates that the tracks' perturbations effect is not felt at the coach case level, as opposed to the bogie and axles where it persists during the coach's circulation. The coach's main suspension acts correspondingly and meets the comfort demands inside the coach.

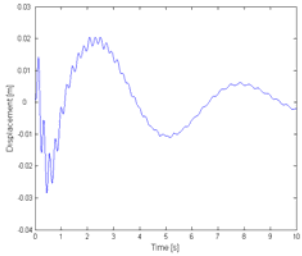

Fig.11 Case lateral displacement

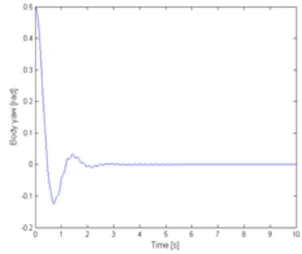

Fig.12 Coach's case yaw

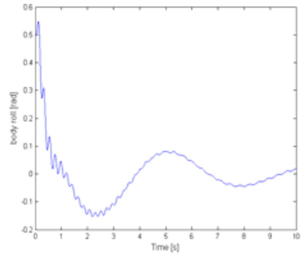

Fig.13 Coach's case roll

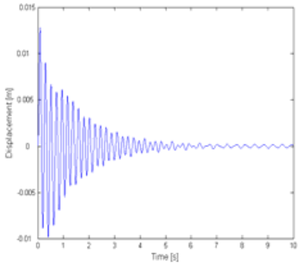

Fig. 14 Bogie's lateral displacement

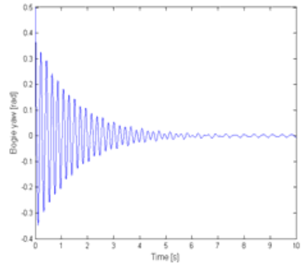

Fig.15 Bogie's yaw

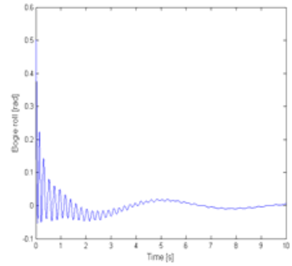

Fig. 16 Bogie's roll

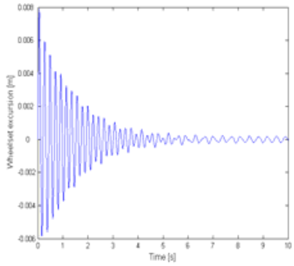

Fig.17 Wheelset's lateral displacement

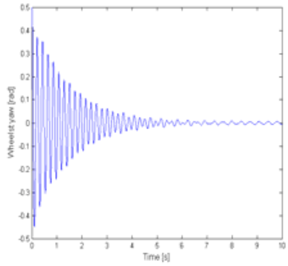

Fig.18 Wheelset's yaw

Table 1- Construction data of the vehicle

Body case mass	m_c= 30760 kg
Bogie mass	m_b= 2300 kg
Wheelset mass	m_o= 1410 kg
Body case moments of inertia	I_{cx}=53596 kgm^2 I_{cz}=1661732 kgm^2
Bogie moments of inertia	I_{bx}= 2240 kgm^2 I_{bz}= 2965 kgm^2
Axles moments of inertia	I_{oy}= 980 kgm^2 I_{ox}=100 kgm^2
Central suspension stiffness	k_{cx}=133 kN/m k_{cy}=133 kN/m k_{cz}=473 kN/m
Axle suspension stiffness	k_{ox}= 256 kN/m k_{oy}= 885 kN/m k_{oz}= 904 kN/m
Central suspension damping	ρ_{cx}= 0 kN/m/s ρ_{cy}= 25 kN/m/s ρ_{cz}= 18 kN/m/s
Damping of the axle suspension	ρ_{oz}=3,67 kN/m/s
Wheel tread radius	r_0=0,460 m
The track's gauge	$2e$=1,435 m
The bogie's wheelbase	$2a$ =2,560 m
The distance between bogies	$2l$ = 17,2 m

The distance between the central suspension's springs	$2d_c$= 2 m
The distance between the axle's suspension springs	$2d_o$= 2 m
The distance case center – central suspension	h_{cc}=1,24m
The distance axles suspension - bogie center	h_{ob}=0,01 m
The distance central suspension - bogie center	h_{cb}=0,06 m
Load on wheel	Q=51250 N
The creepage coefficient	χ=190
The spin creepage coefficient	χ_s=0,83
The effective wheel conicity	γ=0,14
The maximum testing speed	v_{max}= 50 m/s

5. THE VEHICLE'S CRITICAL SPEED

The critical speed is the speed where the vehicle becomes unstable due to the fact that, on the wheel – rail contact, the creepage becomes pure slip [1]. The vehicle's maximum circulation speed must be lower than the critical speed. The equations system describing the vehicle's movement is considered as a continuous dynamic system in time. The internal stability of this type of system solely depends on the distribution of the eigenvalues of the characteristic matrix in the complex plan. The coach's critical speed is determined using the construction characteristics of the coach model seen above and the movement equations given by (26). We proceed then in calculating the eigenvalues of the characteristic matrix of the system resulted through the variable change:

$$\{y\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} \tag{37}$$

that has the form:

$$\{\dot{y}\} = [E]\{y\} + \{F^*(t)\} \tag{38}$$

with

$$[E]= \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \tag{39}$$

The dynamic system is asymptotically stable if and only if all the eigenvalues of the matrix E have a negative real part. Determining the eigenvalues of the matrix E was accomplished in MATLAB using the specific command and increasingly varying the coach's circulation speed. As long as the real part of all the eigenvalues obtained is negative – the coach's movement is stable. If detecting a speed value for which at least one determined eigenvalue has the real part positive the speed's variation step is refined up to the necessary precision.

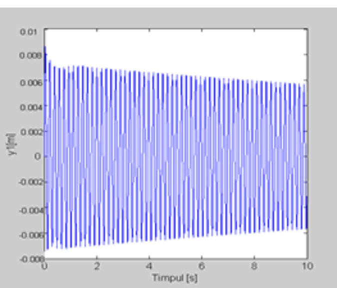

Fig.19 Axle lateral displacement , v<230 km/h

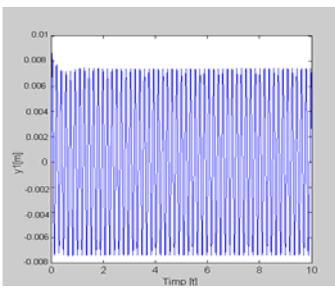

Fig.20 Axle lateral displacement , v=230 km/h

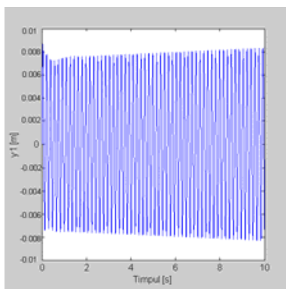

Fig.21 Axle lateral displacement, v>230 km/h

In what regards the coach used in the simulation, the critical speed was determined at a value of 63,7 m/s (~ 230 km/h). In a loaded state, the coach's critical speed will increase as a result of the stabilizing centering effect. The coach's response is presented in the fig. 19-21 - the lateral displacement of the first axle - at inferior, equal and superior speeds to the critical value.

6. THE RAILWAY VEHICLE SUSPENSION STUDY

The mathematical model determined in the previous chapter can be used to improve the design of railway vehicles. Thus, applying the eigenvalues method, the influence of several construction characteristics of the vehicle over the critical speed can be studied.

The axles' suspension construction holds a particular importance over the vehicle stability on a horizontal plan. In general, a growth in the axles' suspension stiffness leads to a significant stability growth.

Thus, if a longitudinal rigidity growth is accomplished, from 250 to 300 kN/m, the vehicle's critical speed can be augmented with up to 18 km/h.

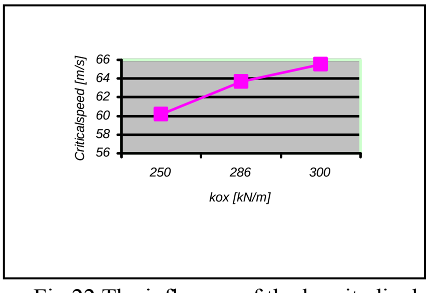

Fig.22 The influence of the longitudinal stiffness of the axle suspension

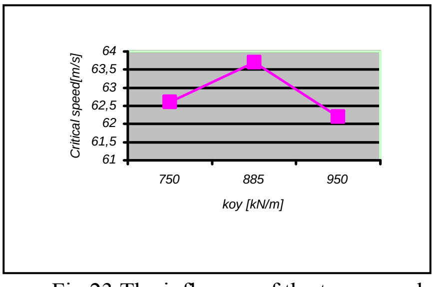

Fig.23 The influence of the transversal stiffness of the axle suspension

In the case of the studied vehicle it was noticed that a maximum critical speed of 64 m/s can be accomplished under the conditions of an axle suspension with a transversal stiffness around 885 kN/m, according to fig. 23.

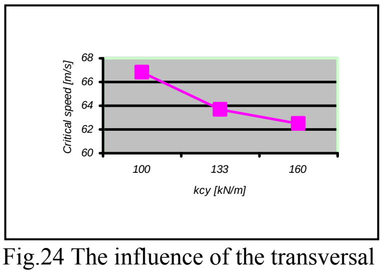

Fig.24 The influence of the transversal stiffness of the central suspension

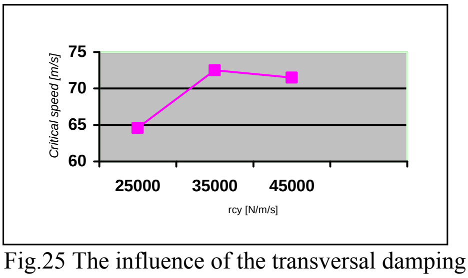

Fig.25 The influence of the transversal damping

Using more and more rigid suspensions also brings an intensification of the wear of the bogies subassemblies. The figure 24 suggests that central suspension stiffness growth on a vertical direction reduces the coach's critical speed. Thus, a vertical stiffness growth of 60% in the central suspension produces a critical speed decrease of 15 km/h. If the coach's central suspension transversal damping value grows from 20 kN/m/s to 30 kN/m/s, the critical speed increases with more than 20 km/h. If the damping has very high values the dampers become very rigid and have the tendency of behaving like bogie – case coupling elements, transmitting oscillations from the rolling apparatus to the coach case, reducing thus both dynamic performance and vehicle comfort.

7. CONCLUSIONS

The article presents a mathematical model with 17 degrees of freedom for a passenger coach reaching a maximum speed of 160 km/h.The model considers the coach's lateral oscillations, respectively the lateral displacement, yawing and rolling motions of the concentrated mass building up the associated multibody model: the coach case, the bogies and the wheelsets.

The equations system describing the vehicle's movement on a lateral direction was solved through numerical methods in order to determine its components' response to the coach's movement on an irregular track – the non – linear component of the equations system. The multibody system based formulation provides a reliable method for the study of the coach's lateral dynamics. The critical speed and the response at 180 km/h of the coach used to exemplify the mathematical model were determined and confirm the ability of the coach to operate at the maximal construction speed.

It was shown that the tracks' perturbations effect is reduced at the coach case level, as opposed to the bogie and axles where it persists during the coach's circulation. The coach's main suspension acts correspondingly and meets the comfort demands inside the coach.

Vehicle performance optimization is possible and it was shown that we obtained an increase of the critical speed with 20 km/h exclusively through an adequate suspension design. However, this undertaking must be the result of an optimization and adequacy process of the suspension's construction parameters in relation with the domain in which the railway vehicle is used and according to the specific operating conditions. Under this extent the presented mathematical model can represent a useful instrument in the calculation, design and optimization of the dynamic performances of railway vehicles.

The presented mathematical model offers developing opportunities considering the non-linearities of the wheel – rail contact and the situations when the vehicle runs in a curve.

REFERENCES

[1] I. Sebesan, *Dinamica vehiculelor feroviare*, Editura Tehnica, 1995.
[2] I. Sebesan, I. Copaci, *Teoria sistemelor elastice la vehiculele feroviare*, Editura Matrix Rom Bucuresti, 2008.
[3] I. Sebesan, D. Hanganu, *Proiectarea suspensiilor pentru vehicule pe sin*e, Editura Tehnica, 1993.
[4] V.K. Garg, R.V. Dukkipati, *Dynamics of railway vehicles systems*, Academic Press Toronto, 1984.
[5] S.Y. Lee, Y.C. Cheng, Hunting stability analysis of high-speed railway vehicle trucks on tangent tracks, *Journal of sound and Vibration*, 2005.
[6] A.H. Wickens, *Fundamentals of rail vehicles dynamics*, Swets& Zeitlinger, 2002.
[7] J.J. Kalker, *Wheel-rail rolling contact theory*, Wear, 1991.
[8] Y.T. Fan, W.F. Wu, Stability analysis of railway vehicles and its verification through field test data, *Journal of Chinese Institute of Engineers*, vol.**29**, 2006.
[9] Y. He, J. McPhee, Optimization of the lateral stability of rail vehicles, *Vehicle System Dynamics*, 2002.
[10] S. D. Iwnicki, A.H. Wickens, *Validation of a Matlab railway vehicle simulation using a scale roller rig*, Manchester Metropolitan University, 2004.
[11] M. Messouci, Lateral stability of rail vehicles – a comparative study, *ARPN Journal of Engineering and applied Sciences*, vol.**4**, 2009**.**
[12] H. True, On the theory of nonlinear dynamics and its applications in vehicle system dynamics, *Vehicle System Dynamics*, 1999.
[13] D.H. Wang, W.H. Liao, Semi-active suspension systems for railway vehicles using magnetorheological dampers, *Vehicle Systems Dynamics*, 2009.
[14] A.A. Shabana, *Dynamics of Multibody Systems*, Cambridge University Press, UK, 2005.