Features of application of the Lanchester-type mathematical models in stochastic formulation when assessing the realities of air-land battle

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Abstract: *This study provides a brief overview of the application of possible modifications of Lanchester-type models, namely, the representation of differential equations of such models in stochastic form. The stochastic setting of differential levels is used in Dynamic models if it is necessary to take into account the influence of random fluctuations (in particular, in radio engineering, thermodynamics, population dynamics models, etc.). As for Lanchester-type models, their stochastic appearance would allow considering the influence of random factors and elements of uncertainty, which are present to a certain extent in any combat operations. At the same time, unlike deterministic models, the numerical solution of systems of stochastic differential equations in such models requires the use of special methods, the choice of a specific one may be based on the requirements for the need to obtain an unambiguous approximate solution, or the probability distribution of the desired quantities. The possibility of obtaining different types of solutions is due to a characteristic feature of the developed methods for numerical integration of stochastic differential equations, namely, the existence of weak and strong approximate methods for solving them. For Lanchester equations, as models for predicting* *the probable course and results of combat operations, it seems appropriate to obtain a solution precisely in the form of parameters for distributions of random variables, which is possible after processing the results of using weak numerical methods. In addition, such methods are considered easier to implement in practice. Of particular note are the issues of estimating the stability of solutions (in the sense of Lyapunov) of stochastic models. While for Lanchester-type models, approximate practical methods for estimating stability can be considered, especially in relation to the simplest, linear statements of basic equations. The study considers an example of using the stochastic Lanchester-type model based on a system of linear inhomogeneous differential equations, with assumptions about the stability of solutions to the stochastic formulation of such equations.*

Key Words: *Lanchester-type models, stochastic differential equations, numerical solution methods, stability of the solution, combat operations.*

1. INTRODUCTION

Interest in Lanchester-type models, which can be traced in publications, in particular [1], [2], [3], [4], [5], [6], [7], [8], as well as the use of such models in modern complexes of combat simulation (such as JCATS and JTLS), indicates the effectiveness of such a device for operational forecasting of the course of combat operations. Thus, in [1], [2], [3] it is noted that even as part of powerful complexes of mathematical models of combat operations, so-called express models are necessary and are widely used. The main purpose of such models is to conduct rapid estimated calculations (without the need to use a significant amount of initial data and computational resources) of the initial balance of forces of the opposing sides, as well as predict the likely course and results of planned actions. It is clear that the methods of analytical modelling are mainly used for such tasks. A special place in this class is occupied by the Lanchester-type models, the advantages of which include a clear physical content of components that can be included in various modifications of equations (which results in the ability to interpret the results obtained in a clear way), the efficiency of obtaining results and control over calculations.

Appropriate directions for modifying models of this type (to increase the adequacy of models by approximating the description of the realities of combat operations with their help) should be considered the use of stochastic components in systems of differential equations in order to take into account the uncertainties of various combat operations. At the same time, simulating the influence of various random factors on the course of combat operations in this way makes it necessary to present them in the form of stochastic differential equations (SDE), for the solution of which numerical methods are known and tested (such as Runge-Kutta methods of various accuracy orders, or modifications of the Euler method) are ineffective.

Methods for solving SDE and their systems differ from the mentioned methods of numerical integration, they have a number of features taking into account which, to obtain acceptable results for Lanchester-type models in the stochastic formulation seems to be quite an urgent task.

Despite the fact that the conventional form of equations was proposed by Lanchester at the beginning of the 20th century, such models are still being further developed theoretically today, which is noted, in particular, in studies [1], [2], [3], [4], [5], [6], [7], [8]. Studies in the field of complex military systems, the results of which are summarised in the above sources, show that modifications of the Lanchester-type models are a fairly effective tool for their qualitative analysis and identification of new patterns of their development, taking into account nonlinear relations between the parameters of systems. However, although in [4] the expediency of such a direction of development of Lanchester models as translating them into the form of SDE was noted, and in [1] various methods for simulating random processes in differential equations were detailed, in these studies, the possibilities of solving SDE using Lanchester equations were rather neglected.

That is why the purpose of this study is to consider certain features of using Lanchester-type models as systems of differential equations in stochastic formulation.

2. MATERIALS AND METHODS

A detailed analysis and classification of Lanchester-type models is given in [6]. Having considered and analysed most of the existing classical productions, the author of [6], [7] comes to a conclusion that such models can be considered to a certain extent "ideal", adapted for idealised deterministic conditions.

Real conflicts will differ, first of all, in time, their duration will be determined by the supply of various resources (human, material) and the influence of random factors. Thus, the study considers an example of one of the simplest variants of the Lanchester-type model, which predicts the restoration of the number of opposing groups:

$$\begin{cases} \dfrac{dx}{dt} = (-c-b)y + d(x(0)-x); \\ \dfrac{dy}{dt} = (-g-f)x + h(y(0)-y), \end{cases} \tag{1}$$

where: *x, y* – the number of groups of opposing sides at a given time $t \geq 0$; $t = 0$, $X(0)$, $in(0)$ – initial conditions for the number of groups at the time of the outbreak of hostilities; *c*, *g* – the intensity of losses due to the impact of one type of weapon (for example, on point objects); *b*, *f* – the intensity of losses due to the impact of another type of weapon (for example, on flat objects); *d, h* –intensity of recovery of the number of warring parties.

The study also considers the most common type of SDE systems. According to [9], [10], a stochastic differential equation is understood as a differential equation in which one or more terms reflect a stochastic process. In this case, the SDE system can be represented as:

$$dx_i(t) = a_i(t)x_i(t)dt + b_i(t)x_i(t)dw(t), t \in \mathbb{R}^+ \tag{2}$$

where: $a_i(t), b_i(t)$ – square matrices of order *N* with continuous and limited coefficients at $t \in \mathbb{R}^+$; $w(t)$– standard Wiener process.

The main properties of the Wiener process are as follows: $w(0) = 0$ with a probability of 1; $w(t)$, $t \geq 0$ is a process with independent increments; $w(t) - w(s) \sim N(0, t-s)$, where $s < t$; the trajectories of the Wiener process are continuous functions of time; the Wiener process is undifferentiated.

As for the last property and type of record (dw), this type is historically determined [9], although it makes no sense from a formal mathematical standpoint. As can be seen from (2), the system consists of ordinary (non-stochastic) differential equations and an additional part describing the stochastic process using Wiener noise (dw). Usually Wiener noise is implemented as: $dw = \varepsilon\sqrt{dt}$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $\varepsilon_i \sim N(0,1)$ – normal distribution with zero mean and unit variance.

The usual deterministic part of the equations forms the wear function, which determines the main direction of movement of the system solution. Accordingly, the stochastic part forms a diffusion function that determines the effect of random fluctuations on the underlying solution of the system.

3. RESULTS AND DISCUSSIONS

With regard to the above version of the Lanchester equation (1), it is appropriate to assume that the main direction of movement of the decision (the current number of parties) will be determined by the influence of the weapons of the opposing parties. Under this assumption, the coefficients of the wear function are assumed, respectively, to intensity *c*, *g*, *b*, *f*. Replenishment of the number of parties is carried out under the influence of fire from the opposite side and a number of other difficult-to-predict factors, respectively, the additive component of the system (1) can be taken as a function of diffusion with coefficients *d, h*. Assuming, for simplicity, that all the coefficients in (1) are constant, in order to translate the equations of such a system into the SDE form, the Black-Scholes formula can be used [10], [12], [13], [14], [15]:

$$dx = axdt + bxdw \quad (3)$$

where: a, b – wear and diffusion coefficients, respectively, the coefficients do not depend on t.

For an example of using form (3) when representing system (1) in stochastic form, it is assumed that the values of all the listed coefficients are known. From a practical standpoint, it should be added that the values of the intensity of losses of the parties and the restoration of their numbers can be known from the experience of using troops, or obtained by simulation modelling. Let the values of the coefficients be: c= 0.0141; g= 0.0137; b = 0.0152; f = 0.0078; d = 0.0302; h = 0,0264. For certain conditions, the base system is represented as:

$$\begin{cases} \dfrac{dx}{dt} = [-0.0141 - 0.0152]y + 0.0302[x(0) - x]; \\ \dfrac{dy}{dt} = [-0.0137 - 0.0078]x + 0.0264[y(0) - y]. \end{cases} \quad (4)$$

Accordingly, taking into account (3), the selected Lanchester equation for the above distribution of wear and diffusion functions is represented as:

$$\begin{cases} \dfrac{dx}{dt} = -0.0293 \cdot y + 0.0302 \cdot [x(0) - x]dw; \\ \dfrac{dy}{dt} = -0.0215 \cdot x + 0.0264 \cdot [y(0) - y]dw. \end{cases} \quad (5)$$

Such a transition would allow considering an important qualitative characteristic of solutions of systems of any differential equations, namely their stability (in the sense of Lyapunov). In general, the following conditions for the existence of stable SDE solutions are formulated [16-18]: a trivial solution of system (2) is exponentially stable in the root-mean-square, if for an arbitrary solution $x_1(t)$ of a system (2) with a non-random initial condition $x_1(t_0), \exists d_1 > 0, d_2 > 0, \forall t, t_0 \in \mathbb{R}^+, t \geq t_0$, is executed:

$$m|x_1(t)|^2 \leq d_1|x_1(t_0)|^2 exp\{-d_2(t - t_0)\} \quad (6)$$

where: m – mathematical expectation.

However, taking into account the transition to the SDE system (5) from the basic linear inhomogeneous equations (4), as well as using the provision [10] on an approximate estimate of the stability of SDE solutions, the following can be accepted for the case under consideration [19], [20], [21], [22]. According to Lyapunov’s First Method, in most cases the stability of nonlinear differential equations can be estimated by linearised equations of the first approximation. The same approach can be used with respect to SDE, as indicated in [10], [23] in the following interpretation: a value judgment about the stability of SDE solutions can be carried out by the stability of the deterministic (non-stochastic) part of the equations. However, the peculiarity of the case under consideration is that the basic systems of equations are known, therefore, the stability assessment of system (4) can be carried out by classical methods (in the form of the roots of the characteristic equation):

$$\begin{bmatrix} -0.0302 - k & -0.0293 \\ -0.0215 & -0.0264 - k \end{bmatrix} \tag{7}$$

roots: $k_1 = -0.0031, k_2 = -0.0535$.

The conditions for the existence of stable solutions allow for the conclusion about the stability of solutions according to system (4) with the selected example of coefficient values. Therefore, using the basic system (4) as equations of the first approximation with respect to the stochastic system (5), it is possible to draw an estimated conclusion about the stability of solutions of the SDE system at the selected coefficients. Root type (k_1, k_2) of a characteristic equation confirming the existence of a stable solution of differential equations: real, negative, different ($k_1 < 0$; $k_2 < 0$; $k_1 \neq k_2$); complex with a negative real part ($k_1 = \alpha + and\beta$; $k_2 = \alpha - and\beta$ ($\alpha < 0$)); purely imaginary ($k_1 = and\beta$; $k_2 = -and\beta$, $k_1 = 0$; $k_2 < 0$, $k_1 = k_2 < 0$) [24], [25], [26].

For a practical example and randomly taken initial conditions: $x(0) = 33184$, $y(0) = 25795$ (the initial number of opposing parties), the solution of system (4) at the time $t = 5$, obtained by numerical integration by the Runge-Kutta method of the 4th order of accuracy, is $x(5) = 29905$, $y(5) = 22629$. Result relative to the value x obtained for similar initial conditions, according to system (5) and using the specialised weak numerical Euler-Maruyama method, the following parameters of the obtained distributions are represented (1000 implementations): $\tilde{m}$ (x) = 28728 with a standard deviation of 232.93. Admittedly, methods with higher convergence (such as the Milstein method or Burrage-Platen methods) can be used to obtain a solution. But in this case, the expediency of using weak numerical methods is emphasised [27], [28], [29].

Strong methods allow finding unambiguous approximate solutions based on a separate implementation, and in some cases this is necessary. In other cases, the goal may be to find the probability distribution of the solution $x(t)$, and individual implementations are not crucial. Weak solutions exist precisely to meet this need. When using Lanchester equations in stochastic form, the mathematical expectations of the desired quantities from the obtained distributions are more representative quantities than their deterministic values. That is, for the given example, the result in the form $\tilde{m}(x) = 28728$ is more trustworthy than $x(5)=29905$. This trust is conditioned by taking into account (albeit to a certain extent) the influence of hard-to-predict factors in the course of military operations, as well as the use of the Fischer maximum likelihood principle when choosing $\tilde{m}(x)$ as a representative characteristic of the value x.

4. CONCLUSIONS

Lanchester-type models and their numerous modifications remain a fairly effective tool for qualitative analysis and identification of new patterns of dynamic military systems, taking into account nonlinear relationships between their parameters. One of the directions of development of this type of models is to bring the differential equations of models to a stochastic form in order to take into account random factors that will affect the course and results of combat operations and which are more or less present in their conduct.

The main features that require attention are: the correct choice of wear and diffusion functions in the compiled stochastic Lanchester-type models (this choice is based on the physical essence of the process reflected in the model); the assessment of the stability of solutions that would be obtained from the compiled SDE, and which can be carried out by evaluating the stability of the basic (non-stochastic) Lanchester equations; the choice as a whole of one of the weak numerical methods of integrating SDE systems (to obtain the distribution parameters of the required values as their representative characteristics under the influence of random factors). Presenting the equations of Lanchester-type models in stochastic form would bring such models closer to the combat realities and help to increase their adequacy.

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