# Control synthesis in the case of cost index with receding horizon

Ioan URSU*[,1], Adrian TOADER[1], Daniela ENCIU[1]
Mihaela DAVIDESCU[1], Cristiana JIDOVU[1]

*Corresponding author
[1]INCAS – National Institute for Aerospace Research “Elie Carafoli”,
220 Bld. Iuliu Maniu, Bucharest 061126, Romania,
ursu.ioan@incas.ro*, toader.adrian@incas.ro, enciu.daniela@incas.ro,
davidescu.mihaela@incas.ro, jidovu.cristiana@incas.ro

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***Abstract:*** *A modified quadratic cost problem for linear systems utilizes receding horizon control (moving terminal constraints) to achieve asymptotic stability and effective feedback stabilization. By modifying the standard linear-quadratic regulator (LQR) cost function, this method produces stabilizing feedback, particularly for time-varying systems where standard algebraic Riccati equations may not apply. An application is proposed for a quarter car model active suspension system.*

***Key Words:*** *LQR, LQG, receding horizon, quarter car model, active suspension*

## 1. INTRODUCTION. RECEDING HORIZON METHOD

Although a possible suggestion of the receding horizon cost index (*receding horizon* in English; *horizon fuyant*, in French) generated by *predictive control* [1], [2] is not excluded, a causal relationship between the two approaches is difficult to establish. Moreover, in the most frequently cited contribution to the field, the work of Kwon and Pearson [3], it is noted that the control law associated with the receding horizon method is actually a generalization of the stabilizing feedback control law obtained by Kleinman [4]. This, on the one hand; on the other hand, the status of the problem can be derived from a bilocal optimization problem with quadratic index [5]. With these clarifications, the receding horizon method is mainly constituted as a stabilization method, but which also has (sub)optimality valences [3]; further details are given in [6], [7], [8].

Let the system and the performance index (with finite horizon) to be minimized be

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{y}_p = \boldsymbol{C}_p\boldsymbol{x},$$
$$\boldsymbol{J}(\boldsymbol{u}) = \int_{t_0}^{t_f} \left(\boldsymbol{y}_p^T\boldsymbol{Q}_J\boldsymbol{y}_p + \boldsymbol{u}^T\boldsymbol{R}_J\boldsymbol{u}\right)\mathrm{d}\boldsymbol{t} + \boldsymbol{x}^T(t_f)\boldsymbol{P}_f\boldsymbol{x}(t_f) \tag{1}$$

with $\boldsymbol{x} \in P^n$, $\boldsymbol{u} \in P^m$, $\boldsymbol{y}_p \in P^p$ the variable matrices $\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}_p(t), \boldsymbol{Q}_J(t) \geq 0$, $\boldsymbol{R}_J(t) > 0$ (hereinafter, the time variable $t$ is omitted), are assumed continuous, with only point discontinuities, and boundary conditions

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_{0,} \quad \boldsymbol{x}(t_f) = 0 \tag{2}$$

It is known that the optimal control law $\boldsymbol{u}$ can be deduced from the Hamiltonian structure

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{p}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\\ -\boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p & -\boldsymbol{A}^T\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}\end{bmatrix} \coloneqq \boldsymbol{H}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{p}\end{bmatrix}, \qquad \begin{bmatrix}\boldsymbol{x}(t_f)\\ \boldsymbol{p}(t_f)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \tag{3}$$

which represents a rewritting of the system, by coupling the Riccati equation

$$\begin{gathered}\boldsymbol{p} = \boldsymbol{P}\boldsymbol{x}, \qquad \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{u} = -\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}\\ -\dot{\boldsymbol{P}} = \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\boldsymbol{P}, \qquad \boldsymbol{P}(t_f) = \boldsymbol{P}_f\end{gathered} \tag{3'}$$

The solution to problem (3) is obtained through the transition matrix, partitioned as follows:

$$\boldsymbol{S}(t,t_0) = \begin{bmatrix}\boldsymbol{\Psi}(t,t_0) & \boldsymbol{\Omega}(t,t_0)\\ \boldsymbol{X}(t,t_0) & \boldsymbol{\Lambda}(t,t_0)\end{bmatrix} \tag{4}$$

**Proposition 1.** *Let the problem be defined by the system (1)–(2), with the cost index to be minimized with respect to* $\boldsymbol{u}$. *a) The open-loop control* $\boldsymbol{u}$ *is given by the law* $\boldsymbol{u} = -\boldsymbol{R}_J^{-1}\boldsymbol{B}^T[\boldsymbol{X}(t,t_0) - \boldsymbol{\Lambda}(t,t_0)\boldsymbol{\Omega}^{-1}(t,t_0)\boldsymbol{\Psi}(t,t_0)]\boldsymbol{x}(t_0)$; *b) The closed-loop control* $\boldsymbol{u}$ *is given by the law* $\boldsymbol{u} = \boldsymbol{R}_J^{-1}\boldsymbol{B}^T\boldsymbol{\Omega}^{-1}(t_f,t)\boldsymbol{\Psi}(t_f,t)\,\boldsymbol{x}$; *c) The closed-loop control* $\boldsymbol{u}$ *can be written as follows:* $\boldsymbol{u} = -\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\boldsymbol{P}^{-1}(t,t_f)\,\boldsymbol{x}$, *where* $\boldsymbol{P}^{-1}(\tau,\sigma)$ *is the inverse of a unique positive-definite symmetric matrix* $\boldsymbol{P}(\tau,\sigma)$, *satisfying Riccati matrix differential equation*

$$\frac{-\partial\boldsymbol{P}(\tau,\sigma)}{\partial\tau} = -\boldsymbol{A}\boldsymbol{P}(\tau,\sigma) - \boldsymbol{P}(\tau,\sigma)\boldsymbol{A}^T - \boldsymbol{P}(\tau,\sigma)\boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p\boldsymbol{P}(\tau,\sigma) + \boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T,$$

$$\tau \le \sigma, \qquad \boldsymbol{P}(\sigma,\sigma) = \boldsymbol{P}_f^{-1}$$

**Proof.**
a) with

$$\boldsymbol{p} = \boldsymbol{X}(t,t_0)\boldsymbol{x}(t,t_0) + \boldsymbol{\Lambda}(t,t_0)\,\boldsymbol{p}(t_0),\ \boldsymbol{x} = \boldsymbol{\Psi}(t,t_0)\boldsymbol{x}(t_0) + \boldsymbol{\Omega}(t,t_0)\,\boldsymbol{p}(t_0) \text{ and } \boldsymbol{x}(t_f) = 0 \Rightarrow$$

$$0 = \boldsymbol{\Psi}(t_f,t_0)\,\boldsymbol{x}(t_0) + \boldsymbol{\Omega}(t_f,t_0)\boldsymbol{p}(t_0) \Rightarrow \boldsymbol{p}(t_0) = -\,\boldsymbol{\Omega}^{-1}(t_f,t_0)\boldsymbol{\Psi}(t_f,t_0)\boldsymbol{x}(t_0) \Rightarrow$$

$$\boldsymbol{p} = [\boldsymbol{X}(t,t_0) - \boldsymbol{\Lambda}(t,t_0)\boldsymbol{\Omega}^{-1}(t_f,t_0)\boldsymbol{\Psi}(t_f,t_0)]\boldsymbol{x}(t_0) \Rightarrow$$

$$\boldsymbol{u}(\boldsymbol{t}) = -\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\,\boldsymbol{p} \quad \text{q.e.d.}$$

$$\text{b) } 0 = \boldsymbol{x}(t_f) = \boldsymbol{\Psi}(t_f,t)\boldsymbol{x}(t) + \boldsymbol{\Omega}(t_f,t)\boldsymbol{p}(t) \Rightarrow \boldsymbol{p} = -\boldsymbol{\Omega}^{-1}(t_f,t)\boldsymbol{\Psi}(t_f,t)\boldsymbol{x} \Rightarrow$$

$$\boldsymbol{u} = \boldsymbol{R}_J^{-1}\boldsymbol{B}^T\,\boldsymbol{p} \quad \text{q.e.d.}$$

c) Given the control expression from point b), it results

$\boldsymbol{P}(t_{f,}t) = -\boldsymbol{\Omega}^{-1}(t_f,t)\boldsymbol{\Psi}(t_f,t)$, thus $\boldsymbol{P}(t,t_f) = \boldsymbol{P}^{-1}(t,t_f) = -\boldsymbol{\Psi}^{-1}(t_f,t)\,\boldsymbol{\Omega}(t_f,t)$ $(*)$

Whereas

$$\frac{d\boldsymbol{S}(t,t_0)}{dt} = \boldsymbol{H}\boldsymbol{S}(t,t_0), \qquad \boldsymbol{S}(t,t_0) = \boldsymbol{S}^{-1}(t_0,t) \Rightarrow$$

$$-\boldsymbol{S}^{-1}(t_0,t)\dot{\boldsymbol{S}}_{\boldsymbol{t}}(t_0,t)\boldsymbol{S}^{-1}(t_0,t) = \boldsymbol{H}\boldsymbol{S}^{-1}(t_0,t) \Rightarrow -\dot{\boldsymbol{S}}_{\boldsymbol{t}}(t_0,t) = \boldsymbol{S}(t,t_0)\boldsymbol{H} \Rightarrow$$

$$-\dot{\boldsymbol{S}}_{\boldsymbol{t}}(t_f,t) = \boldsymbol{S}(t_f,t)\boldsymbol{H} \Leftrightarrow$$

$$-\begin{bmatrix}\dot{\boldsymbol{\Psi}}(t_f,t) & \dot{\boldsymbol{\Omega}}(t_f,t)\\ \dot{\boldsymbol{X}}(t_f,t) & \dot{\boldsymbol{\Lambda}}(t_f,t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Psi}(t_f,t) & \boldsymbol{\Omega}(t_f,t)\\ \boldsymbol{X}(t_f,t) & \boldsymbol{\Lambda}(t_f,t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\\ -\boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p & -\boldsymbol{A}^T\end{bmatrix}.$$

Writting the relation (*) as $\boldsymbol{P}(\tau,\sigma) = -\boldsymbol{\Psi}^{-1}(\sigma,\tau)\boldsymbol{\Omega}(\sigma,\tau)$, with $\tau \leq \sigma$ and deriving, we get

$$-\frac{\partial \boldsymbol{P}(\tau,\sigma)}{\partial\tau} = \frac{\partial\left[\boldsymbol{\Psi}^{-1}(\tau,\sigma)\boldsymbol{\Omega}(\tau,\sigma)\right]}{\partial\tau} = \dot{\boldsymbol{\Psi}}_{\boldsymbol{\tau}}^{-1}(\tau,\sigma)\boldsymbol{\Omega}(\tau,\sigma) + \boldsymbol{\Psi}^{-1}(\tau,\sigma)\dot{\boldsymbol{\Omega}}_\tau(\tau,\sigma) =$$

$$-\boldsymbol{\Psi}^{-1}(\tau,\sigma)\,\dot{\boldsymbol{\Psi}}_\tau(\tau,\sigma)\boldsymbol{\Psi}^{-1}(\tau,\sigma)\boldsymbol{\Omega}(\tau,\sigma) + \boldsymbol{\Psi}^{-1}(\tau,\sigma)\dot{\boldsymbol{\Omega}}_\tau(\tau,\sigma) =$$

$$-\boldsymbol{\Psi}^{-1}(\tau,\sigma)\left[-\boldsymbol{\Psi}(\tau,\sigma)\boldsymbol{A} + \boldsymbol{\Omega}(\tau,\sigma)\boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p\right]\boldsymbol{\Psi}^{-1}(\tau,\sigma)\boldsymbol{\Omega}(\tau,\sigma) +$$

$$+\boldsymbol{\Psi}^{-1}(\tau,\sigma)\left[\boldsymbol{\Psi}(\tau,\sigma)\boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T + \boldsymbol{\Omega}(\tau,\sigma)\boldsymbol{A}^{\boldsymbol{T}}\right] =$$

$$= \boldsymbol{A}\boldsymbol{\Psi}^{-1}(\tau,\sigma)\,\boldsymbol{\Omega}(\tau,\sigma) + \boldsymbol{\Psi}^{-1}(\tau,\sigma)\,\boldsymbol{\Omega}(\tau,\sigma)\boldsymbol{A}^{\boldsymbol{T}} -$$

$$-\boldsymbol{\Psi}^{-1}(\tau,\sigma)\,\boldsymbol{\Omega}(\tau,\sigma)\boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p\boldsymbol{\Psi}^{-1}(\tau,\sigma)\,\boldsymbol{\Omega}(\tau,\sigma) + \boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T. \quad \text{q.e.d.}$$

The basic idea of the stabilizing feedback control (associated with a receding horizon cost index) is clarified by the following observations:

a) From (1) it follows that, by imposing $\boldsymbol{P}_\text{f} = \infty$, the control is required to force the state of the system to tend to zero in time. Since a transfinite value is not operable in numerical calculations, Proposition 1 shows how the standard Riccati control equation is manipulated to use the final boundary condition, and in the control calculation the inverse of the solution of the Riccati equation thus modified is used.

b) In order for the stabilizing effect (with a suboptimal minimization effect, in the subsidiary) on state $\boldsymbol{x}$ to be even more pronounced, the horizon, defined by the integration limits of the cost index (1), is modified in the sense of what is called a receding horizon, namely $(t, t + t_\text{p})$, with insignificant effects on the formulation of the entire Proposition 1: only the substitutions $t_0 \to t$, $t_\text{f} \to t + t_\text{p}$ are involved.

c) In the general case of the time-varying system, the solution (with receding horizon) $\boldsymbol{P} := \boldsymbol{P}(t, t + t_\text{p})$ of the Riccati equation is unimplementable, since it requires an instantaneous "backward" integration, on the interval $[(t, t + t_\text{p})]$, with the current $t$. On the other hand, for the case of the time-invariant system and for constant matrices $\boldsymbol{Q}_J$ and $\boldsymbol{R}_J$, an implementable control is obtained, since the Riccati equation

$$-\dot{\boldsymbol{P}} = \boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\boldsymbol{T}} + \boldsymbol{P}\boldsymbol{C}_p^T\boldsymbol{Q}_J\boldsymbol{C}_p\boldsymbol{P} - \boldsymbol{B}\boldsymbol{R}_J^{-1}\boldsymbol{B}^T, \qquad \boldsymbol{P}(0) = 0 \tag{5}$$

integrated on interval $[0, t_\text{p}]$ provides a computable solution *offline* $\boldsymbol{P}(t, t_\text{p})$ (and $\boldsymbol{u} = -\boldsymbol{R}_J^{-1}\boldsymbol{B}^T\boldsymbol{P}^{-1}$). It is clear that the respective solution does not depend on the receding interval $[t, t + t_\text{p}]$, translatable with the end at the origin (in $[0, t_\text{p}]$). In addition, for simplicity, the sense of integration has been reversed, from 0 to $t_\text{p}$.

d) The parameter $t_\text{p}$ of the method, which can be called, in short, the receding horizon method, can be interpreted as a *virtual forecast time* [7].

e) For the asymptotic stability of the closed-loop system with the control obtained by the receding horizon method, it is sufficient that the pair $(\boldsymbol{A}, \boldsymbol{B})$ be stabilisable ($\boldsymbol{A}$, $\boldsymbol{B}$ – constant matrices).

## 2. NUMERICAL APPLICATION AND CONCLUSIONS

The analysis performed in this section is mainly carried out in the time domain and is expressed in statistical values – std or rms – of the *performance output* components in the presence of natural-random disturbances induced by the roadway.

The performances of the various systems are assessed according to the size of the reduction in the influence of these disturbances on the performance output. The performance indicators as such, expressed as mentioned in std or rms, are the following: comfort, denoted C – the variation of the (absolute) vertical acceleration of the suspended mass, safety, denoted S – the variation of the relative displacement unsprung mass-roadway, and the variation of the suspension travel (geometric indicator), denoted G.

It should be noted that the first indicator is *subjective* (there are standards or authors that pay attention to the influence of celerity, or maximum acceleration, on the human body), and the other two are *objective*.

The rms measurement of C is quite widely accepted [9]. The stochastic processes that define the response of most of the models considered – linear – being normal and of zero mean, in principle the two measures, rms and std, are identical.

To avoid possible confusion, it should be noted that *having good performance means having low values of these indicators, C, S and G.*

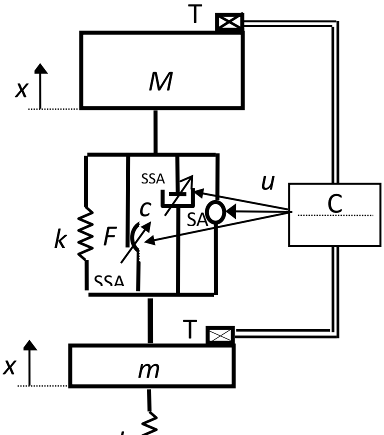


Fig. 1 – Quarte car model with 2 DoF

The legend of Fig. 1 is completed with some notations: *F* – dry friction force; SSA – semiactive servomechanism; SA – active servomechanism; C – controller; *u* – control; T – traductor.

In [11], the aim was to obtain a *common denominator of these results*, by relating all mathematical models of suspension systems and control synthesis applications to a single input data system (see Fig. 2 and [10]):

**1**) *body geometry*: $a$ = 1.22 m, $b$ = 1.22 m, $d'$ = 0.49 m, $d''$ = 0.49 m, $E$ = 1.31 m; **2)** *inertial elements*: $m$ = 1150 kg, $m'$ = 31.5 kg, $m''$ = 70 kg, $I_\alpha$ = 913 kgm$^2$, $I_\beta$ = 212 kgm$^2$, $I_{\beta''}$ = 22 kgm$^2$; **3)** *suspension elements*: $k'$ = 11 370 N/m, $c'$ = 835 Ns/m, $k''$ = 14 100 N/m, $k'_0$= 167

000 N/m, $k_0''$= 218 000 N/m, $c''$ = 1 685 Ns/m; **4)** *servomechanism data*: $S$ = 10 cm2, $k_{SV}$ = 50 $cm^3$/(smA), $k_c$ = 30 / 8 000 $cm^5$/daN, $k_{Qp}$ = 2.5 $cm^5$/(daNs); **5)** *reduced model* (with 2 GL): $M$ = 1 150 / 4 kg, $m$ = 31.5 kg, $k_1$ = 11 370 N/m, $k_2$ = 167 000 N/m, $c_1$ = 835 Ns/m.

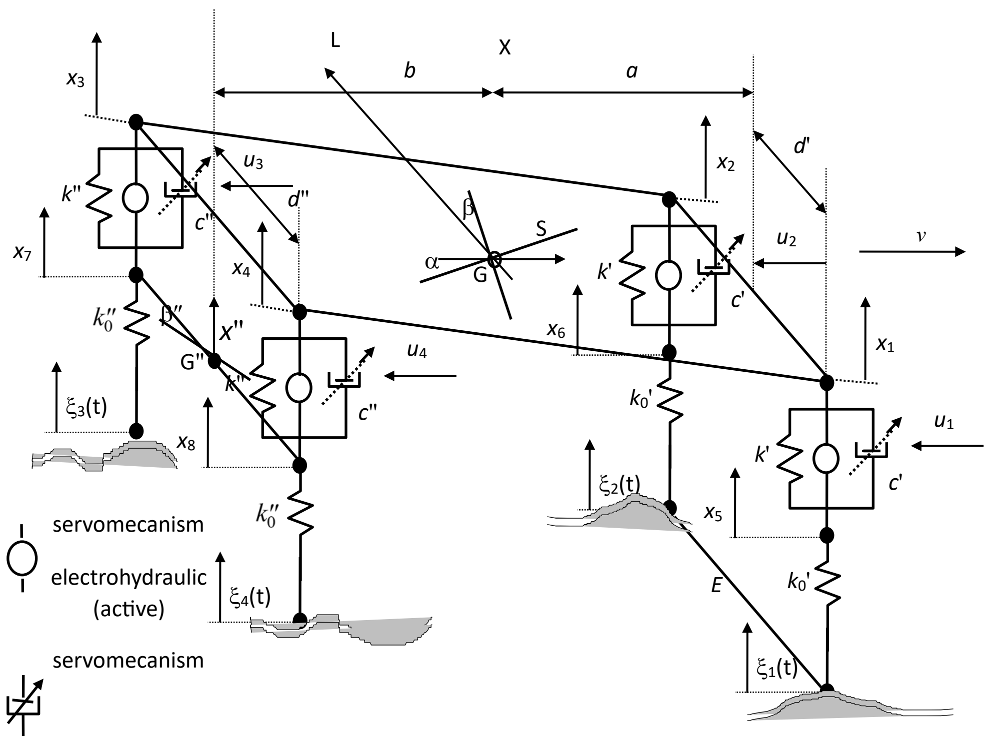


Fig. 2 – Physical model of suspension system with 7 DoF

SLX – Cartesian coordinate system centered at G, the center of gravity of the total suspended mass of the vehicle;

G″ – center of gravity of the unsprung mass of the rear axle;

$E$ – ampatament;

$[\xi_1(t) \dots \xi_4(t)]^T$ – the vector stochastic process that describes the effect of roadway disturbances on the vehicle;

$\{x(t)\}$ – the stochastic process that describes the vertical displacement of the center of gravity G of the total suspended mass of the vehicle relative to the static equilibrium position;

$\{x''(t)\}$ – the stochastic process that describes the vertical displacement of the center of gravity G″ relative to the static equilibrium position;

$[x_1(t) \dots x_4(t)]^T$ – the vector stochastic process that describes the vertical displacements of suspended masses relative to the static equilibrium position;

$[x_5(t) \dots x_8(t)]$ – the vector stochastic process that describes the vertical displacements of the wheel centers relative to the static equilibrium position;

$\{\alpha(t)\}$ – the stochastic process describing the pitch angle of the total suspended mass;

$\{\beta(t)\}$ – the stochastic process describing the roll angle of the total suspended mass;

$\{\beta''(t)\}$ – the stochastic process describing the roll angle of the unsprung mass of the rear axle;

$m$ – total suspended mass of the vehicle;

$m', m''$ – unsprung masses: of the front axle, with independent suspensions, respectively, of the rear axle, with solid suspension;

| | |
|---|---|
| $I_\alpha$ | – moment of inertia of the suspended mass about the transverse axis passing through the center of gravity G; |
| $I_\beta$ | – the moment of inertia of the suspended mass about the longitudinal axis passing through the center of gravity G; |
| $I_{\beta''}$ | – moment of inertia of the unsprung mass of the rear axle around the longitudinal axis passing through the center of gravity $G''$; |
| $k', k''$ | – the stiffness coefficients of the front and rear axle suspensions, respectively; |
| $k'_0, k''_0$ | – the stiffness coefficients of the front and rear tires, respectively; |
| $c', c''$ | – the damping coefficients of the front and rear suspensions, respectively; |
| $c'_0, c''_0$ | – the damping coefficients of the front and rear tires, respectively; |
| $\left[p_1(t)\dots p_4(t)\right]^{\mathrm{T}}$ | – stochastic processes that describe pressure drops in hydraulic servomechanisms; |
| $S$ | – the active surface of hydraulic servomechanisms; |
| $[i_1(t) \dots i_4(t)]^T$ | – the vector stochastic process that describes the control currents of servo valves; |
| $k_u$ | – servovalve current-flow amplification; |
| $\{Q(t)\}$ | – the stochastic process that describes the supply flow of the servovalves; |
| $k_{Q_p}$ | – flow-pressure drop amplification of the generalized flow characteristic of hydraulic servomechanisms; |
| $V$ | – the volume of the working chamber for the average piston position; |
| $B_l$ | – the compressibility modulus of the working fluid; |
| $\mathrm{P}^{m\times n}$ | – the real $m \times n$ -dimensional linear space of matrices; |
| $\mathrm{P}^n$ | – the linear space of $n$-dimensional real vectors; |
| $\boldsymbol{A}^T$ | – transpose of a matrix $\boldsymbol{A}$; |
| $\mathbf{0}^{m\times n}, \boldsymbol{I}^{m\times n}$ | – null and identity matrices $m \times n$, $m \times m$, respectively; |
| $t$ | – the independent variable (time), relative to which the derivatives of the dependent variables are written; usually, the writing of $t$ is omitted; |
| $\dot{x}(t)$ | – derivative with respect to time of the variable $x(t)$. |

**The synthesis of the estimator** in the receding horizon method can follow a strategy similar to the synthesis of the control, as Thomas [12] does. In the numerical application, however, the standard LQG estimator (6) was used

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{K_E}(\boldsymbol{y_0} - \boldsymbol{C}_0\hat{\boldsymbol{x}}) \tag{6}$$

having the gain matrix

$$\boldsymbol{K_E} := \overline{\boldsymbol{Q}}\boldsymbol{C}_0^T\boldsymbol{R}_\eta^{-1} \tag{7}$$

built on the basis of the unique and symmetric solution $\overline{\boldsymbol{Q}} \geq 0$ of the the algebraic matrix Riccati estimation equation

$$\boldsymbol{AQ} + \boldsymbol{QA^T} - \boldsymbol{QC_0^T R_\eta^{-1} C_0 Q} + \boldsymbol{DQ_w D^T} = \boldsymbol{0} \tag{8}$$

(since this also ensures the asymptotic estimation of the state of the controlled system, under the conditions of Theorem 5.2 [13] defined in section 5.1.1).

For the numerical application summarized in this section, the following mathematical model variant will be considered (see also Fig. 1):
The defined subvariant, with 5 states, is as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{D}\boldsymbol{w}, \qquad \boldsymbol{x} = [x_2 - \xi \quad \dot{x}_2 \quad x_1 - x_2 \quad \dot{x}_1 \quad F]^{\boldsymbol{T}}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k_2}{m} & -\frac{c_1}{m} & \frac{k_1}{m} & \frac{c_1}{m} & \frac{1}{m} \\ 0 & -1 & 0 & 1 & 0 \\ 0 & \frac{c_1}{M} & -\frac{k_1}{M} & -\frac{c_1}{M} & -\frac{1}{M} \\ 0 & 0 & 0 & 0 & -\frac{1}{\tau} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{k_F}{\tau} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

The model of a servomechanism containing a force tracking loop with a time constant is involved τ

$$\tau\dot{F} + F = k_F u \tag{10}$$

Performance output is $y_p = [\ddot{x}_1 \quad x_1 - x_2 \quad x_2 - \xi]^T$, and with three measured output variants: suspension travel $x_1 - x_2$, acceleration $\ddot{x}_1$, or relative speed $\dot{x}_1 - \dot{x}_2$; then

$$\boldsymbol{C}_p = \begin{bmatrix} 0 & \frac{c_1}{M} & -\frac{k_1}{M} & -\frac{c_1}{M} & -\frac{1}{M} \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the corresponding matrices $\boldsymbol{C}_0$ are obtained accordingly.

The active synthesis application of the receding horizon (OG) control of the performance index is summarized in Fig. 3 (the corresponding numerical values are presented in Table 1; what is of particular interest is the evaluation of the influence of the virtual preview time $t_p$. Three variants of the measured output are considered: suspension travel, acceleration or relative velocity.

Two variants of control selection are presented below, one active, being suggestive as such (Table 2), and the other semi-active (Table 2), successively from the LQG control, then from the control based on the virtual lead time tp; these applications refer to the system (8'). In all the measured output variants, the specific values of the parameters were $R_\eta = Q_w / 100$, $Q_w = 0{,}54$ cm$^2$/s, $R_J = 1$, and for the servomechanism $\tau = 0{,}01$ s and $k_F = 1$ daN/mA. The values of the $Q_J$ matrix are recorded below the Tables. Some special parameters are also indicated in the cases of selection (semiactive or exclusively active): $n_{act}$ – the number of steps (out of the total of 6 000) in which, in the calculation performed by simulating the active system with the lsim subroutine in MATLAB, an active type operation was attested; $n_0$, $n_k$, $nc$ – the number of steps in which the system with the control selection was monitored on the null portion, on the bisector branch, respectively, on the saturation zone of the selection characteristic represented in fig. 6.2 [13] (it was considered $c_{max} = 1.5$ daNs/cm and $c_{min} = 0$); $n_i$ – the number of steps T after which control is launched.

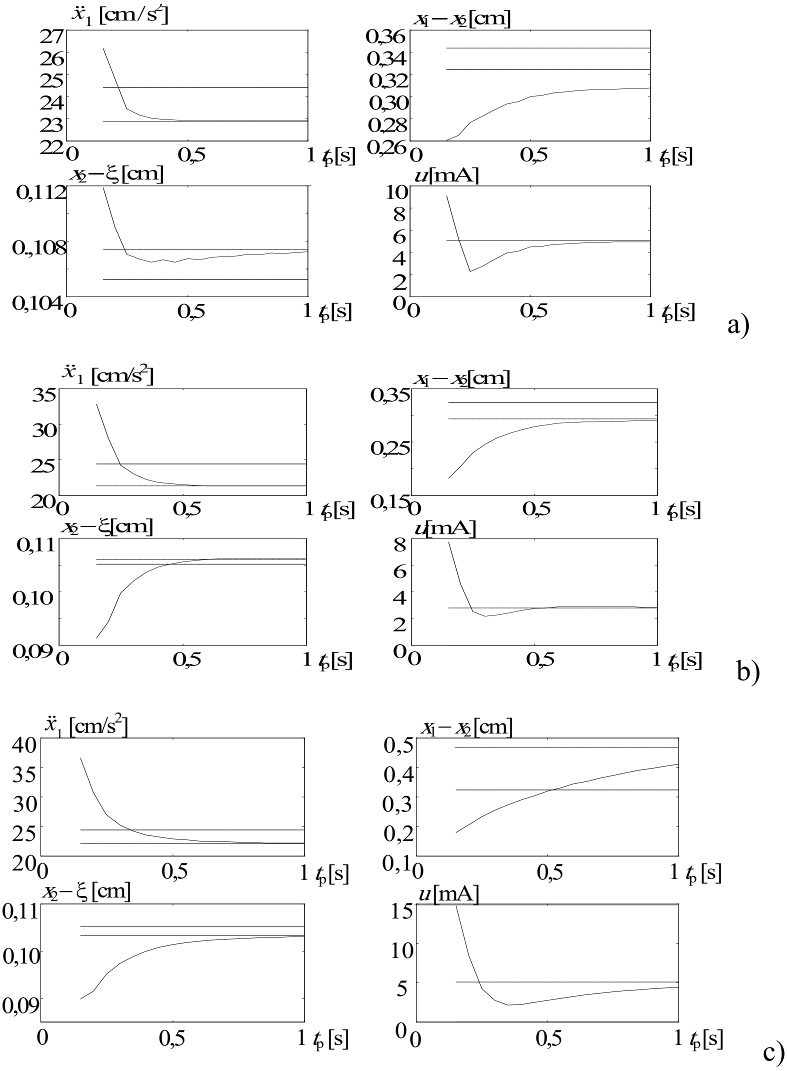


Fig. 3 – Receding horizon method (RH). Influence of virtual forecast time $t_p$ [$s$] on the performance of the active system; cases: ......... passive; ------- LQG ; – RH

a) $y_0 = x_1 - x_2 + \eta$; b) $y_0 = \ddot{x}_1 + \eta$; c) $y_0 = \dot{x}_1 - \dot{x}_2 + \eta$.

A comment is necessary regarding the significance of semiactive selection: by virtue of the relation (4.6') [13], [14] the synthesized control $u$ is measured in V (or mA) – therefore, it is not directly a force, as the semiactive selection algorithm assumes. Taking into account the coefficient $k_F$ (chosen to be unitary), the control $u$ represents a force in value and the procedure remains valid in fact. Two methods of calculating the performances are recorded, by the Liapunov equation of covariance of states (which provides "exact" results) and by integration with the MATLAB subroutine lsim. In the latter case, a Gaussian white noise sample with length N = 6 000 was introduced, with the sampling step T = 0.005 s (it is advisable that T be even smaller, T = 2π/(100x2πη), η being the maximum frequency of the system, $\eta = \sqrt{k_2/m}$; the „scaling" correction (by dividing the samples by the square root of the step) for a narrowband white noise is used for both state and measure noise

From the analysis of the results, three conclusions can be drawn: a) since the difficulty of simultaneously improving the two indicators C and S is already known, it is noticeable that the RH method favors this goal (Table 1); b) in this case too, the semiactive system appears to be competitive in terms of overall performance with the active one; c) a virtual prediction time of about 0.4 s is sufficient to achieve the performance of the method.

Table 1. Control synthesis using the receding horizon method (RH)
a) Active synthesis: dependence of indicators C, G, S on virtual preview time and measured output

| | a) $y_0 = x_1 - x_2 + \eta$ | | | | b) $y_0 = \ddot{x} + \eta$ | | | | c) $y_0 = \dot{x}_1 - \dot{x}_2 + \eta$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_p$ [s] | $\ddot{x}_1$ [cm/s²] | $x_1 - x_2$ [cm] | $x_2 - \xi$ [cm] | $u$ [mA] | $\ddot{x}_1$ [cm/s²] | $x_1 - x_2$ [cm] | $x_2 - \xi$ [cm] | $u$ [mA] | $\ddot{x}_1$ [cm/s²] | $x_1 - x_2$ [cm] | $x_2 - \xi$ [cm] | $u$ [mA] |
| passive | 24,43 | 0,32 | 0,11 | 0 | 24,43 | 0,32 | 0,11 | 0 | 24,43 | 0,32 | 0,11 | 0 |
| active | 22,90 | 0,34 | 0,11 | 5,08 | **21,34** | **0,29** | **0,11** | **2,78** | **22,09** | **0,46** | **0,10** | **5,08** |
| 0,05 | 550,18 | 0,36 | 0,31 | 1 501 | 836,19 | 0,48 | 0,48 | 961,5 | 689,7 | 0,41 | 0,42 | 1274 |
| 0,20 | 24,80 | 0,27 | 0,11 | 5,27 | 27,98 | 0,20 | 0,09 | 4,59 | 30,79 | 0,21 | 0,09 | 8,42 |
| **0,40** | 22,96 | 0,29 | 0,11 | 3,96 | **21,86** | **0,27** | **0,10** | **2,46** | 23,59 | 0,29 | 0,10 | 2,23 |
| 0,50 | 22,93 | 0,30 | 0,11 | 4,48 | 21,50 | 0,28 | 0,11 | 2,73 | 22,93 | 0,32 | 0,10 | 2,76 |
| **0,70** | 22,91 | 0,31 | 0,11 | 4,83 | 21,32 | 0,29 | 0,11 | 2,91 | **22,42** | **0,36** | **0,10** | **3,65** |
| 1,00 | 22,91 | 0,31 | 0,11 | 4,94 | 21,31 | 0,29 | 0,11 | 2,84 | 22,20 | 0,41 | 0,10 | 4,42 |
| | $\boldsymbol{Q}_J = \mathrm{diag}(10^2, 10^3, 10^5)$, $R_J = 1$ | | | | $\boldsymbol{Q}_J = \mathrm{diag}(1, 1, 5\times10^4)$, $R_J = 1$ | | | | $\boldsymbol{Q}_J = \mathrm{diag}(17, 1, 10^6)$, $R_J = 1$ | | | |

b) Active selection of active control (in case of measured output – acceleration)

| case | $\ddot{x}_1$[cm/s²] | $x_1 - x_2$ [cm] | $x_1 - \xi$ [cm] | $u$ [mA] | observations |
|---|---|---|---|---|---|
| passive | 24,43 | 0,32 | 0,11 | 0 | exact values |
| | 23,22 | 0,36 | 0,10 | 0 | integration `lsim` |
| LQG | **21,34** | **0,29** | **0,11** | **2,78** | exact values |
| | 20,34 | 0,27 | 0,10 | 2,46 | integration `lsim` ($n_{act}$ = 3 037 din 6 000) |
| selection | 20,44 | 0,29 | 0,10 | 4,62 | $\boldsymbol{n_i = 4}$; $n_0$ = 439; $n_k$ = 990; $n_c$ = 71 |
| active (sa) | 20,33 | 0,28 | 0,10 | 4,05 | $\boldsymbol{n_i = 2}$; $n_0$ = 823; $n_k$ = 1 988; $n_c$ = 189 |
| OG | **21,86** | **0,27** | **0,10** | **2,46** | valori exacte |
| ($t_p$ = 0,4 s) | 20,88 | 0,25 | 0,10 | 2,21 | integration `lsim` ($n_{act}$ = 3 165 out of 6 000 ) |
| (idem, sa) | 20,97 | 0,25 | 0,10 | 4,97 | $\boldsymbol{n_i = 4}$; $n_0$ = 462; $n_k$ = 967; $n_c$ = 71 |
| | 20,73 | 0,25 | 0,10 | 3,92 | $\boldsymbol{n_i = 2}$; $n_0$ = 871; $n_k$ = 1961; $n_c$ = 168 |

c) Semiactive selection of active control (in the case of measured output – relative suspension speed)

| case | $\ddot{x}_1$ [cm/s²] | $x_1 - x_2$ [cm] | $x_1 - \xi$ [cm] | $u$ [mA] | observations |
|---|---|---|---|---|---|
| passive | 24,43 | 0,32 | 0,11 | 0 | exact values |
| | 23,22 | 0,31 | 0,10 | 0 | lsim integration |
| LQG | **22,09** | **0,46** | **0,10** | **5,08** | exact values |
| | 21,05 | 0,30 | 0,10 | 2,77 | integration `lsim` ($n_{act}$ = 3 298 din 6 000) |
| selection | 22,35 | 0,35 | 0,10 | 14,00 | $\boldsymbol{n_i = 4}$; $n_0$ = 487; $n_k$ = 919; $n_c$ = 94 |
| semiactive | 21,86 | 0,33 | 0,10 | 14,65 | $\boldsymbol{n_i = 2}$; $n_0$ = 922; $n_k$ = 1882; $n_c$ = 196 |
| OG | **22,42** | **0,36** | **0,10** | **3,65** | exact values |
| ($t_p$ = 0,7 s) | 21,38 | 0,28 | 0,09 | 2,30 | integration `lsim` ($n_{act}$ = 3 406 din 6 000) |
| (idem, | 22,40 | 0,29 | 0,10 | 10,08 | $\boldsymbol{n_i = 4}$; $n_0$ = 448; $n_k$ = 977; $n_c$ = 75 |
| ssa) | 21,90 | 0,28 | 0,10 | 10,57 | $\boldsymbol{n_i = 2}$; $n_0$ = 847; $n_k$ = 1 995; $n_c$ = 158 |

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