Optimal control of propellant consumption during vertical lifting of rocket in homogeneous atmosphere using regularized solution of integral equation of the first kind

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Abstract: *The article is devoted to the study of the optimal control of propellant consumption during vertical lifting of rocket in homogeneous atmosphere using regularized solution of integral equation of the first kind. The problem of lifting of a rocket into desired height along optimal trajectory in the view of minimal consumption of propellant leads to solving the set of differential and integral equations. Problem of optimal control of propellant consumption during lifting of rocket in homogeneous atmosphere is solved using regularized solution of integral equation of the first kind which is solution of corresponding Euler equation on discrete time net. Influence of the regularization parameter and some additional parameters on precision of discreted problem is investigated. Considered algorithm is summed up easily to the case of non-homogeneous atmosphere by introducing dependence of the ballistic coefficient on altitude of flight and to problem of putting spacecraft into determined orbit and suborbital flights by setting desired altitude and velocity and modifying of motion equations.*

Key Words: *ballistics, inverse problems, regularization method, variation principle, Euler equation*

1. INTRODUCTION

A problem of the trajectory optimization of a rocket or a spacecraft with a rocket engine belongs to a class of the dynamic systems optimization problems. Its solution leads to searching for the local or global extremum of a beforehand defined functional determined on the set of the solutions of the controlled dynamic system satisfying some conditions [1], [2], [3]. Applying some restrictions to rocket we have some formulation of the optimization problem [4], [5], [6], [7], [8], [9], [10]. It is well known that its solution is found with the maximum principle by Pontryagin transferring the optimization problem to the boundary problem [9], [10], [11], [12], [13].

There are two models of a rocket engine performance. The first of them matches the non-controlled engine when the reactive force and the relative velocity of exhaust gazes are considered to be constant [14], [15], [16], [17]. The engine just can be turned on or off. That is the most realistic model. The second of them matches the ideal limited power engine when the power of the engine is constant. Under this restriction we can vary the reactive force and the exhaust velocity [14]. In this work we vary both the reactive force and the power of the rocket engine by varying the consumption of a propellant and keeping the exhaust gases velocity. The optimal control problem is to find the trajectory corresponding to the minimal consumption of a propellant.

A problem of vertical lifting of a rocket to desired height with minimal consumption of propellant is solved in [18]. Vertical movement of a body with variable mass $m(\tau)$ in homogeneous atmosphere is described by the next equations [18]:

$$\left\{\begin{aligned} &\frac{dv}{d\tau} = \frac{1}{m(\tau)}[aw(\tau) - cv^2(\tau)] - g, \\ &\frac{dm}{d\tau} = -w(\tau) \end{aligned}\right\} \tag{1}$$

with the initial conditions

$$m(0) = m_0, v(0) = \Upsilon_0 \tag{2}$$

where $\mu \le m(\tau) \le m_0$ is the variable mass of a rocket with propellant, kg; μ is the mass of construction of a rocket, kg; $v(\tau)$ is the velocity of a rocket; $w(\tau)$ is the control function equal to the consumption of propellant trough one second, kg/s; $a = const = 2500\, m/s$ is the relative velocity of exhaust gases; $\mathrm{c} = 0.2 \cdot 10^{-7}\, kg/m$ is the generalized ballistic coefficient of air; $g = 9.81\, m/s^2$ is the free fall acceleration.

The optimal control function $\widetilde{w}(\tau)$ must be positive at a time interval $0 \le \tau \le \mathrm{T}_1$ and equal to zero at an interval $\tau > \mathrm{T}_1$. Gradual decrease of the consumption of propellant begins at the time instant $\tau = 0$ when the velocity is equal Υ_0.

The optimal control function $\widetilde{w}(\tau)$ and parameters corresponding to it are desired: $\mathrm{T}_1 = const$ is the time instant at which burning of propellant is stopped, s; $\Upsilon_1 = v(\mathrm{T}_1) = const$ is the velocity of rocket at the instant T_1, m/s.

2. SOLUTION USING INTEGRAL EQUATION OF THE FIRST KIND

Approximate solution of this problem has a strong singularity at the initial instant of the time, therefore it is unstable and the problem is ill-conditioned [18]. If T_2 is the instant of the time at which the velocity becomes equal to zero and $\Upsilon_2 = v(\mathrm{T}_2) = 0$ then the height into which a rocket is lifted is equal [18]:

$$h = h[v(w)] = \int_0^{T_2} v(\tau)d\tau \tag{3}$$

The height h is summarized by two terms [18]:

$$h = h_1 + \Delta h = \int_0^{T_1} v(\tau)d\tau + \frac{\mu}{2c} ln\left(1 + \frac{\Upsilon_1^2 c}{\mu g}\right) \tag{4}$$

The functional to minimize is [18]:

$$f[\mathrm{w}, \Upsilon_1, \mathrm{T}_1] = 1 - \frac{h[v(w)]}{h_0} \tag{5}$$

where h_0 is a number near to desired maximal height, m. Minimizing the functional (5) is unstable, therefore the regularization method is used.

The stabilizing functional $\Omega[w]$ is applied to find approximate (regularized) solution [18]:

$$\Omega[w] = \int_0^{T_1} \varphi(w) d\tau \tag{6}$$

Then seeking the optimal control $\widetilde{w}(\tau)$ leads to minimizing the smoothing functional [18]

$$M[w_\gamma, \Upsilon_1, \mathrm{T}_1] = f[w_\gamma, \Upsilon_1, \mathrm{T}_1] + \gamma\Omega[w] \tag{7}$$

under the additional conditions:

$$w(\tau) \geq 0 \tag{8}$$

$$\int_0^{T_1} w(\tau) d\tau = m_0 - \mu \tag{9}$$

To find approximate regularized solution we use the next algorithm. Keeping the regularization parameter $\gamma > 0$ we define a consequence of couples of the numbers $\left\{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right\}$. For each such a couple the function $w_\gamma^{(n)}$ minimizing the flattening (smoothing) functional is found [18]:

$$M\left[w_\gamma^{(n)}, \Upsilon_1, \mathrm{T}_1\right] = \inf_{w^{(n)}} M[w^{(n)}, \Upsilon_1, \mathrm{T}_1] \tag{10}$$

Then from the sequence of couples of the numbers $\left\{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right\}$ such a couple $\left\{\Upsilon_1^{(m)}, \mathrm{T}_1^{(m)}\right\}$ is found on which the flattening functional reaches its minimum [18]:

$$M\left[\mathrm{w}_\gamma^{(m)}, \Upsilon_1^{(m)}, \Upsilon_1^{(m)}\right] = \inf_{\Upsilon_1^{(n)}, T_1^{(n)}} M\left[\mathrm{w}_\gamma^{(n)}, \Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right] \tag{11}$$

As a result we get the functions $\mathrm{w}_\gamma^{(m)}, \Upsilon_1^{(m)}, \mathrm{T}_1^{(m)}$ which are considered to be approximate regularized solution of the problem of optimal control. Let us modify the algorithm using regularization of solution of integral equation of the first kind. This equation follows from the relation (4) and is in the velocity $\upsilon(\tau)$ for each couple of the numbers $\left\{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right\}$:

$$\int_0^{T_1} \upsilon(\tau) d\tau \equiv h_1 = h - \Delta h \approx h_0 - \frac{\mu}{2c} ln\left(1 + \frac{\Upsilon_1^2 c}{\mu g}\right) \tag{12}$$

with the boundary conditions $\upsilon(0) = \Upsilon_0, \upsilon(\mathrm{T}_1) = \Upsilon_1$.

From the set of equations (1) we get the differential equation for the varying mass $m(\tau)$ which is connected with the velocity $\upsilon(\tau)$:

$$\frac{dm(\tau)}{d\tau} + \frac{1}{a}\left[\frac{d\upsilon(\tau)}{d\tau} + g\right] m(\tau) + \frac{c}{a}\upsilon^2(\tau) = 0 \tag{13}$$

with the initial condition $m(0) = m_0$.

The consumption of the mass of propellant is found from the same set of equations as

$$w(\tau) = -\frac{dm(\tau)}{d\tau} \tag{14}$$

The procedure to find the optimal consumption of propellant is analogous to the previous one. Keeping the height h_0 we define a consequence of couples of the numbers $\left\{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right\}$.

For each such a couple we solve the integral equation of the first kind (12) for the velocity $\upsilon(\tau)$ using the regularization method.

The function $m(\tau)$ is found for the function $\upsilon(\tau)$ from the equation (13) and the consumption of the propellant $w(\tau)$ from the equation (14). Then from couples of the numbers $\left\{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right\}$ such a couple $\left\{\Upsilon_1^{(m)}, \mathrm{T}_1^{(m)}\right\}$ is found on which the propellant consumption (14) reaches its minimum:

$$w^{(m)}\left[\Upsilon_1^{(m)}, \mathrm{T}_1^{(m)}\right] = \inf_{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}} w^{(n)}\left[\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right] \tag{15}$$

As a result we get the functions $\mathrm{w}_\gamma^{(m)}, \Upsilon_1^{(m)}, \mathrm{T}_1^{(m)}$ which are considered to be approximate regularized solution of the problem of optimal control.

For each couple of the numbers $\left\{\Upsilon_1^{(n)}, \mathrm{T}_1^{(n)}\right\}$ the right-hand side of the equation (12) is put approximately, and $\Upsilon_1^{(n)} \in \left[\Upsilon_1^{(0)}, \Upsilon_1^{(N)}\right]$, where $N = m + r, r \geq 0$. The integral equation:

$$\upsilon \equiv \int_0^{T_1} K(\Upsilon_1, \tau)\upsilon(\tau)d\tau = u_\delta(\Upsilon_1), \Upsilon_1 \in \left[\Upsilon_1^{(0)}, \Upsilon_1^{(N)}\right] \tag{16}$$

has the kernel $K(\Upsilon_1, \tau) = 1$ and the function

$$u_\delta(\Upsilon_1) = h_0 - \frac{\mu}{2c} ln\left(1 + \frac{\Upsilon_1^2 c}{\mu g}\right) \tag{17}$$

The required approximate (regularized) solution of the equation (16), $Av = u_\delta$, is the function $\upsilon_\gamma(\tau)$ which is the solution of the integro-differential equation of Euler [18]. If F_1 is a set of the functions $\upsilon(\tau)$ continuous on the interval $[0, \mathrm{T}_1]$ and having the first order derivatives $d\upsilon(\tau)/d\tau$ square integrable on $[0, \mathrm{T}_1]$, then for the functions $\upsilon(\tau) \in F_1$ the stabilizing functional is determined as [18]:

$$\Omega[\upsilon] = \int_0^{T_1} \left\{q(\tau)\upsilon^2(\tau) + p(\tau)\left(\frac{d\upsilon}{d\tau}\right)^2\right\} d\tau \tag{18}$$

where $q(\tau)$, $p(\tau)$ are defined nonnegative functions such that for every $\tau \in [0, \mathrm{T}_1]$ we have $q^2(\tau) + p^2(\tau) \neq 0$ and $p(\tau) \geq p_0 > 0$ where p_0 is a number.

Let us choose one of these functionals. Minimizing the functional (18) is a conditional extremum problem.

Let us solve it by the method of undetermined Lagrange multipliers; that is let us find the function $\upsilon_\gamma(\tau)$ minimizing the smoothing functional [18]:

$$M^\gamma[\upsilon, u_\delta] = \rho_{L_2}^2(A\upsilon, u_\delta) + \gamma\Omega[z] \tag{19}$$

where [18]

$$\rho_{L_2}(u_1, u_2) = \left\{\int_c^d [u_1(x) - u_2(x)]^2 dx\right\}^{\frac{1}{2}} \tag{20}$$

This is an unconditional extremum problem, in which the regularization parameter is determined from the equation

$$\rho_{L_2}(Av, u_\delta) = \delta \tag{21}$$

with the solution $\gamma = \gamma(\delta)$ depending on the discrepancy δ.

The parameter γ may be determined both by the discrepancy (21) and other ways. Minimizing the stabilizer (18) we have the next Euler equation [18]:

$$\int_0^{T_1} \bar{K}(\tau,t)v(t)dt + \gamma\left\{q(\tau)v(\tau) - \frac{d}{d\tau}\left(p(\tau)\frac{dv}{d\tau}\right)\right\} = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} K(\Upsilon_1,\tau)u_\delta(\Upsilon_1)\ d\Upsilon_1 \tag{22}$$

where $K(\Upsilon_1,\tau) = 1$,

$$\bar{K}(\tau,t) = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} K(\Upsilon_1,\tau)K(\Upsilon_1,t)\ d\Upsilon_1 = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} d\Upsilon_1 = \Upsilon_1^{(N)} - \Upsilon_1^{(0)} \tag{23}$$

$$\begin{gathered}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} K(\Upsilon_1,\tau)u_\delta(\Upsilon_1)\ d\Upsilon_1 = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}}\left[h_0 - \frac{\mu}{2c}ln\left(1+\frac{\Upsilon_1^2 c}{\mu g}\right)\right]\ d\Upsilon_1 = \\ = h_0\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right) - \frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} ln\left(1+\frac{\Upsilon_1^2 c}{\mu g}\right)d\Upsilon_1\end{gathered} \tag{24}$$

$$\begin{gathered}\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right)\int_0^{T_1} v(t)dt + \gamma\left\{q(\tau)v(\tau) - \frac{d}{d\tau}\left(p(\tau)\frac{dv(\tau)}{d\tau}\right)\right\} = \\ = h_0\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right) - \frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} ln\left(1+\frac{\Upsilon_1^2 c}{\mu g}\right)\ d\Upsilon_1\end{gathered} \tag{25}$$

This equation is solved with one of the boundary conditions following from the equality to zero of the solution or its first derivative on the bounds of the interval $[0, \mathrm{T}_1]$:

$$v(0) = 0, v(\mathrm{T}_1) = 0 \tag{26}$$

$$v(0) = 0, v'(\mathrm{T}_1) = 0 \tag{27}$$

$$v'(0) = 0, v\ (\mathrm{T}_1) = 0 \tag{28}$$

$$v'(0) = 0, v'\ (\mathrm{T}_1) = 0 \tag{29}$$

If $v(0) = \Upsilon_0$, $v(T_1) = \Upsilon_1$, where Υ_0, Υ_1 are known numbers, then passing on to the function $\tilde{v}(\tau)$ by the formula [18]

$$\begin{gathered}v(\tau) = \tilde{v}(\tau) + \frac{\Upsilon_0}{T_1}(T_1 - \tau) + \frac{\Upsilon_1}{T_1}\tau = \tilde{v}(\tau) + \Upsilon_0 - \frac{\Upsilon_0}{T_1}\tau + \frac{\Upsilon_1}{T_1}\tau = \\ = \tilde{v}(\tau) + \Upsilon_0 + \frac{1}{T_1}(\Upsilon_1 - \Upsilon_0)\tau\end{gathered} \tag{30}$$

we get the equation for $\tilde{v}(\tau)$ with the same kernel but other right-hand side the solution of which (with corresponding right-hand part) satisfies the boundary conditions $\tilde{v}(0) = 0$, $\tilde{v}(\mathrm{T}_1) = 0$.

We can find approximate solution for $\tilde{v}(\tau)$ from the Euler equation [1], [2] (with transformed right-hand side) satisfying to the boundary conditions $\tilde{v}(0) = 0$, $\tilde{v}(\mathrm{T}_1) = 0$:

$$\begin{aligned}\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)\int_0^{T_1}&\left[\tilde{v}(t)+\Upsilon_0+\frac{1}{T_1}(\Upsilon_1-\Upsilon_0)t\right]dt+\gamma\times\\ &\times\left\{q(\tau)\left[\tilde{v}(\tau)+\Upsilon_0+\frac{1}{T_1}(\Upsilon_1-\Upsilon_0)\tau\right]\right.\\ &\left.-\frac{d}{d\tau}\left(p(\tau)\frac{d}{d\tau}\left[\tilde{v}(\tau)+\Upsilon_0+\frac{1}{T_1}(\Upsilon_1-\Upsilon_0)\tau\right]\right)\right\}\\ &=h_0\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)-\frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}}ln\left(1+\frac{\Upsilon_1^2c}{\mu g}\right)\,d\Upsilon_1\end{aligned}\tag{31}$$

We have

$$\begin{aligned}\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)&\left[\int_0^{T_1}\tilde{v}(t)dt+\Upsilon_0T_1+(\Upsilon_1-\Upsilon_0)\frac{T_1}{2}\right]\\ &++\gamma\left\{q(\tau)\left[\tilde{v}(\tau)+\Upsilon_0+\frac{1}{T_1}(\Upsilon_1-\Upsilon_0)\tau\right]\right.\\ &\left.-\frac{d}{d\tau}\left(p(\tau)\left[\frac{d\tilde{v}(\tau)}{d\tau}+\frac{1}{T_1}(\Upsilon_1-\Upsilon_0)\right]\right)\right\}=\\ &=h_0\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)-\frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}}ln\left(1+\frac{\Upsilon_1^2c}{\mu g}\right)\,d\Upsilon_1\end{aligned}\tag{32}$$

We put $q(\tau)=q=const>0, m$; $p(\tau)=p=const>0, m\cdot s^2$. Then:

$$\begin{aligned}\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)&\left[\int_0^{T_1}\tilde{v}(t)dt+\Upsilon_0T_1+(\Upsilon_1-\Upsilon_0)\frac{T_1}{2}\right]\\ &+\gamma\left\{q\left[\tilde{v}(\tau)+\Upsilon_0+\frac{1}{T_1}(\Upsilon_1-\Upsilon_0)\tau\right]-p\frac{d^2\tilde{v}(\tau)}{d\tau^2}\right\}=\\ &=h_0\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)-\frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}}ln\left(1+\frac{\Upsilon_1^2c}{\mu g}\right)\,d\Upsilon_1\end{aligned}\tag{33}$$

and having been left in the left-hand side the terms including the desired function $\tilde{v}(\tau)$ we got the equation

$$\begin{aligned}\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)&\int_0^{T_1}\tilde{v}(t)dt+\gamma q\tilde{v}(\tau)-\gamma p\frac{d^2\tilde{v}(\tau)}{d\tau^2}=\\ &=\left(\Upsilon_1^{(N)}-\Upsilon_1^{(0)}\right)\left[h_0-\frac{T_1}{2}(\Upsilon_0+\Upsilon_1)\right]-\gamma q\left[\Upsilon_0+\frac{\tau}{T_1}(\Upsilon_1-\Upsilon_0)\right]\\ &-\frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}}ln\left(1+\frac{\Upsilon_1^2c}{\mu g}\right)\,d\Upsilon_1\end{aligned}\tag{34}$$

where $\gamma>0$ is the dimensionless regularization parameter; $q>0, m$; $p>0, m\cdot s^2$ are positive dimension quantities.

Let us write down a difference analogue of the equation (34) on a uniform net with the increment $\Delta\tau$. We divide the interval $[0, T_1]$ into M equal parts and set the ends of got intervals as nodes of the net [18]:

$$\tau_i = i\Delta\tau, i = 1,2 \dots, M, \Delta\tau = \frac{\mathrm{T}_1}{M} \tag{35}$$

Replacing the integral in the left-hand side of the equation (34) by the integral sum corresponding to it according to the formula of rectangles, for example, and $\upsilon''(\tau)$ by corresponding difference expression, we have [18]

$$\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right)\Delta\tau \sum_{j=1}^{M} \tilde{\upsilon}_j + \gamma q \tilde{\upsilon}_i + \gamma p \frac{2\tilde{\upsilon}_i - \tilde{\upsilon}_{i-1} - \tilde{\upsilon}_{i+1}}{\Delta\tau^2} = f_i, i = 1,2, \dots, M \tag{36}$$

$$f_i = f(\tau_i) = \left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right)\left[h_0 - \frac{T_1}{2}(\Upsilon_0 + \Upsilon_1)\right] - \gamma q \left[\Upsilon_0 + \frac{\tau_i}{T_1}(\Upsilon_1 - \Upsilon_0)\right] - \frac{\mu}{2c}\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} ln\left(1 + \frac{\Upsilon_1^2 c}{\mu g}\right) d\Upsilon_1 \tag{37}$$

The values of the right-hand side f_i are calculated analytically or numerically. At the same time the numbers N, M of the net points on the coordinates Υ_1, τ are independent. If $i = 1, i = M$ then there undefined values $\tilde{\upsilon}_0$ and $\tilde{\upsilon}_{M+1}$ are in the set of linear algebraic equations (37) for the vector $\tilde{\boldsymbol{\upsilon}} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2, \dots, \tilde{\upsilon}_M)$. To satisfy to the boundary conditions we put $\tilde{\upsilon}_0 = \tilde{\upsilon}_1 = 0$ and $\tilde{\upsilon}_{M+1} = \tilde{\upsilon}_M = 0$. Thus the problem of searching for approximate (regularized) solution of the equation (16), $A\upsilon = u_\delta$, leads to solving the set of linear algebraic equations (37) for the vector $\tilde{\boldsymbol{\upsilon}} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2, \dots, \tilde{\upsilon}_M)$ and further passing on to the vector $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \dots, \upsilon_M)$ by the formula (30). It is possible to use, for example, the method of square root or Voevodin's method to solve it [18].

3. CALCULATION OF THE REGULARIZATION PARAMETER

Calculations have been carried out to solve a problem of searching for the optimal consumption of propellant during lifting of a rocket to the height $h_0 = 10\ km$ for the couple of the numbers $\{\Upsilon_1 = 100\ m/s, \mathrm{T}_1 = 100\ s\}$ when the numbers are able to vary in the limits $\pm 10\%$. If the regularization parameter $\gamma = 1$ then a rocket reaches the predetermined height h_0 at which the velocity becomes equal to zero within an accuracy of solution of the set of algebraic equations in a wide range of the quantities q, p. The regularization parameter varies, the height of lifting of a rocket deviates from h_0 keeping values of couple of the numbers $\{\Upsilon_1, T_1\}$. For example, a vector of the regularization parameters $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_M)$ corresponding to the exponential distribution of the velocity according to the law $\upsilon_0(\tau) = \Upsilon_1[1 - \exp(-\tau/\tau_{0.63})]$, where $\tau_{0.63} = \mathrm{T}_1/5\tau$ is the time constant, s, leads to the velocity of lifting of a rocket $h = 8573\ m$ instead of $h \approx h_0 = 10^4 m$ (Fig. 1a).

The quantities q, p vary keeping the regularization parameter $\gamma = 1$, the distribution of the velocity in the time deviates for predetermined couple of the numbers $\{\Upsilon_1, \mathrm{T}_1\}$ leading to the desired velocity profile (Fig. 1a). Simultaneously, the consumption of propellant necessary to lift a rocket into the required height $h \approx h_0 = 10^4 m$ varies a little about 1% of the consumption when $q = 1\ m, p = 1\ m \cdot s^2$. Comparison of the consumption of propellant for different profiles of the velocity shows the exponential distribution of the velocity to be not optimal for the couple of the numbers $\{\Upsilon_1, T_1\}$ as we reach the least consumption when $\gamma = 1$; $q = 0.1\ m, p = 10\ m \cdot s^2$ (Figs. 1b, 1c).

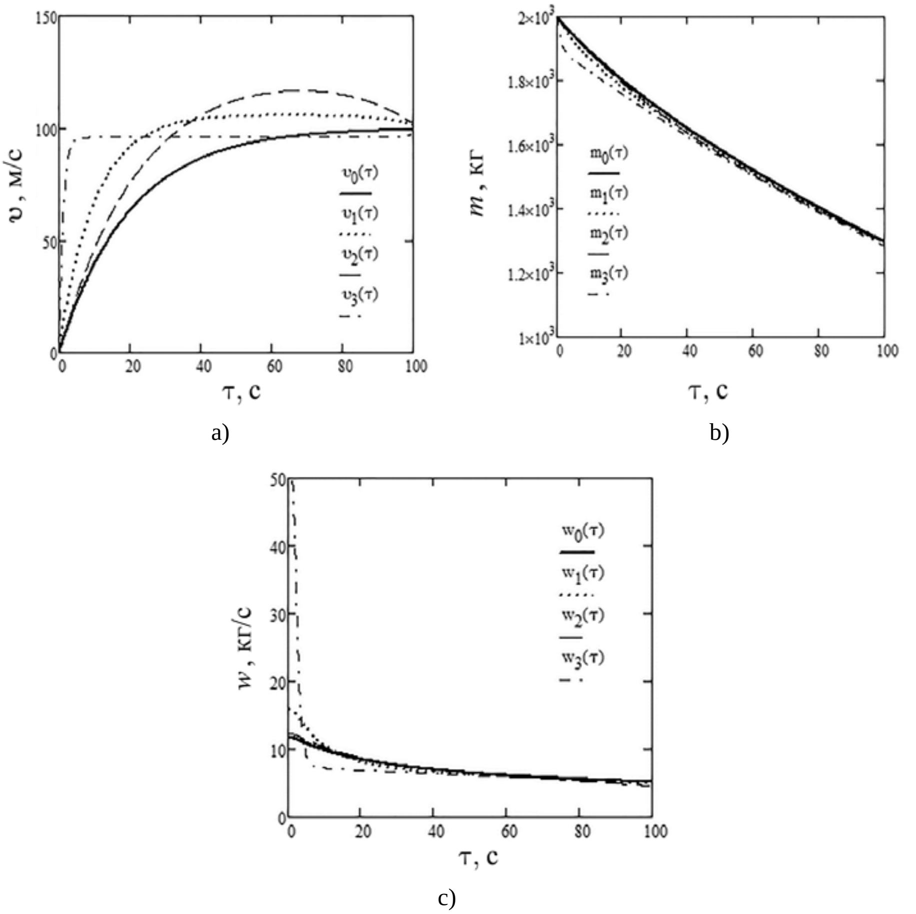

Fig. 1 – Distribution of the velocity (a), the mass of a rocket with propellant (b) and the consumption of propellant (c) in the time (the mass of an empty rocket $\mu = 10^3 kg$, the mass of propellant $\Delta m = 10^3 kg, \Upsilon_1 = 100\ m/s, \mathrm{T}_1 = 100\ s$): $\upsilon_0(\tau)$ is by the exponent, $\gamma = \gamma(\tau)(h = 8473\ m, m(0) - m(\mathrm{T}_1) = 703.3\ kg)$; $\upsilon_1(\tau)$ is by $\gamma = 1, q = 0.1, p = 10$ ($h = 10060$ m$, m(0) - m(\mathrm{T}_1) = 705.6\ kg$); $\upsilon_2(\tau)$ is by $\gamma = 1, q = 0.1, p = 50$ ($h = 10070$ m$, m(0) - m(T_1) = 704.9\ kg$); $\upsilon_3(\tau)$ is by $\gamma = 1, q = 1, p = 1$ ($h = 10020$ m$, m(0) - m(T_1) = 716.0\ kg$)

Let us set a sequence of couples of the numbers $\{\Upsilon_1, T_1\}$ on the intervals $\Upsilon_1 \in [90, 110] m/s$; $\mathrm{T}_1 \in [90, 110] m/s$ and find the least of the propellant consumptions which are calculated for each pair keeping $\gamma = 1$; $q = 0.1\ m$; $p = 10\ m \cdot s^2$. The least propellant consumption $650.2\ kg$ is reached for the couple of the numbers $\{\Upsilon_1 = 90\ m/s, \mathrm{T}_1 = 90\ s\}$ and supplies lifting of a rocket into the height $9975\ m$ for the start mass $2000\ kg$ (Table 1).

Table 1. – The integral propellant consumption equal to the difference $m(0) - m(\mathrm{T}_1)$ and the height reached for couples of the numbers $\{\Upsilon_1, \mathrm{T}_1\}$ (the mass of an empty rocket $\mu = 10^3 kg$, the mass of propellant $\Delta m = 10^3 kg$)

$\Upsilon_1, m/s$; T_1, s	90	95	100	105	110
90	650.2 kg 9975 m	675.9 kg 9972 m	701.2 kg 9969 m	726.0 kg 9966 m	750.3 kg 9963 m
95	652.2 kg 10020 m	678.2 kg 10020 m	703.4 kg 10010 m	728.1 kg 10010 m	752.4 kg 10010 m
100	654.8 kg	680.4 kg	705.6 kg	730.3 kg	754.5 kg

	10070 m	10070 m	10060 m	10060 m	10060 m
105	657.2 kg 10120 m	682.7 kg 10120 m	707.8 kg 10110 m	732.4 kg 10110 m	756.6 kg 10110 m
110	659.5 kg 10170 m	685.0 kg 10170 m	710.0 kg 10170 m	734.5 kg 10160 m	758.7 kg 10160 m

For the conditions of the problem having been solved the desired height is reached when the regularization parameter is equal to $\gamma = 1$. The acceleration has maximum at the instant of time $\tau = 0$ and as a rule it is less than $10\ m/s$ that is admissible for a manned flight as the resulting acceleration is less than $2\ g$. Nevertheless, in the case of lifting of a rocket into greater height the start acceleration should magnify. Then we have to supply some desired distribution of the velocity in time:

$$v_0^{(0)} = v^{(0)}(\tau_0) = \Upsilon_0 \tag{38}$$

$$v_i^{(0)} = v^{(0)}(\tau_i),\ (i = 1,2,\ldots, M\text{-}1) \tag{39}$$

$$v_M^{(0)} = v^{(0)}(\tau_M) = \Upsilon_1 \tag{40}$$

As according to (26)

$$\tilde{v}(\tau) = v(\tau) - \Upsilon_0 - \frac{1}{T_1}(\Upsilon_1 - \Upsilon_0)\tau \tag{41}$$

Then

$$\tilde{v}_i^{(0)} = v_i^{(0)} - \Upsilon_0 - \frac{1}{T_1}(\Upsilon_1 - \Upsilon_0)\tau_i \tag{42}$$

with $\tilde{v}_0^{(0)} = 0$, $\tilde{v}_M^{(0)} = 0$. We have to take into consideration that distribution of the regularization parameter in time affects the height h reached by a rocket: if $\gamma \neq 1$ then $h \neq h_0$.

Considering the difference between desired and determined distributions of the velocity in time to be known (a vector $\boldsymbol{\delta}$) we write down

$$\left\|\boldsymbol{v} - \boldsymbol{v}^{(0)}\right\| = \delta \Rightarrow v_i - v_i^{(0)} = \delta_i (i = 1, 2, \ldots, M) \tag{43}$$

or according to (26)

$$\left\|\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{(0)}\right\| = \delta \Rightarrow \tilde{v}_i - \tilde{v}_i^{(0)} = \delta_i (i = 1, 2, \ldots, M) \tag{44}$$

In problems of control it is natural to suppose $\delta = 0$ leading desired distribution of the velocity to determined one, if it is supplied by smoothness of defined function. Then on the basis of (30) a vector of the regularization parameters is able to be calculated analytically or numerically (Fig. 2). Such a vector of the regularization parameters supplies the velocity distribution equivalent predetermined one (40) that is substituted into the ordinary differential equation (13) for the mass $m(\tau)$ of a rocket. But predetermined function is possible not smooth enough and has not mathematically correct equivalent calculated on the equation (30). In that case we have to find a vector of the regularization parameters supplying stable solution of the equation (30) and minimizing the discrepancy of the velocities (43). To do it we can calculate analytically, use the simple iteration method or the iteration-variation method [19], [20], [21] (Fig. 2).

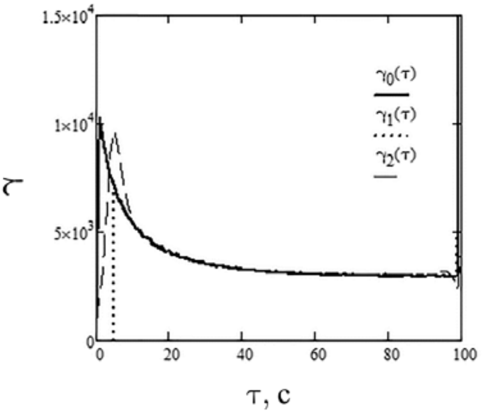

Fig. 2 – Distribution of the regularization parameter $\gamma = \gamma(\tau)$ in the time for the exponential distribution of the velocity in the time $\upsilon_0(\tau)(h = 8573\ m)$: $\gamma_0(\tau)$ is by direct calculation; $\gamma_1(\tau)$ is calculated by the method of simple iteration; $\gamma_2(\tau)$ is calculated by the iteration-variation method

4. CONCLUSIONS

The problem of lifting of a rocket into desired height along optimal trajectory in the view of minimal consumption of propellant leads to solving the set of differential and integral equations. The ordinary differential equation is in the mass of a rocket from the time keeping the free fall acceleration, the ballistic coefficient of atmosphere, the velocity of exhaust gases from a rocket engine and the velocity of a rocket in the time. The integral equation of the first kind is got from mechanics, $\upsilon(\tau) = dh(\tau)/dh, => \upsilon(\tau)d\tau = dh(\tau)$, connecting the velocity of a rocket with the height of lifting. This equation is solved by the regularization method transforming it into the Euler equation which is discredited on the time net as the set of linear algebraic equations in the velocity dependent on the time.

In the right-hand part of the integral equation there is the height corresponding to the velocity of a rocket at the instant when burning of propellant is stopped or the height when the velocity is equal to zero after movement along ballistic trajectory. In the second case there is the second term in the right-hand part of the integral equation. If the desired height is known (with some inaccuracy) then to solve the integral equation in the velocity we have to define couples of the variables: the time of propellant burning and the velocity desired at the time instant when burning is stopped. Also we have to know admissible interval of varying of the velocity at the time instant when a rocket engine is turned off.

In the Euler equation there are regularization parameter which is constant or another function of the time $\gamma = \gamma(\tau) > 0$, the functions $q = q(\tau) > 0\ m$; $p = p(\tau) > 0\ m \cdot s^2$. Keeping $\gamma = 1$ (or a constant) and varying the functions q, p we control the velocity distribution in the time to supply the desired height of lifting of a rocket with accuracy of the solution of the approximate Euler equation. If $\gamma \neq 1$ (or a constant) then the height of lifting of a rocket deviates from the predetermined one. Particularly the regularization parameter is able to be found from predetermined distribution of the velocity as a function of the time. A problem of lifting of a rocket along optimal trajectory is important both to calculate a flight of multi-stage rocket, when we put the height corresponding to the velocity of a rocket at the time instant when propellant burning is stopped, and for suborbital flights, when we put the height corresponding to stopping of lifting after movement along ballistic trajectory. If the heights are great then we have to set the function of the ballistic coefficient of air resistance depending on the height dividing a trajectory into intervals on which the coefficient is to be constant approximately. Putting the ballistic coefficient of atmosphere and the free fall acceleration of another planet we can use this algorithm to calculate optimal trajectory of takeoff of a rocket from them. Also we can modify the algorithm using different coordinate set to write down the equation of the motion.

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